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NOTICE TO AUTHORS

1. *Communications.*—Papers must be communicated to the Society by a Fellow. They should be accompanied by a summary at the *beginning* of the paper conveying briefly the content of the paper, and drawing attention to important new information and to the main conclusions. The summary should be intelligible in itself, without reference to the paper, to a reader with some knowledge of the subject; it should not normally exceed 200 words in length. **Authors are requested to submit MSS. in duplicate. These should be typed using double spacing and leaving a margin of not less than one inch on the left-hand side. Corrections to the MSS. should be made in the text and not in the margin.** Unless a paper reaches the Secretaries more than seven days before a Council meeting it will not normally be considered at that meeting. By Council decision, MSS. of accepted papers are retained by the Society for one year after publication; unless their return is then requested by the author, they are destroyed.

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A. Corlin, *Zs. f. Astrophys.*, **15**, 239, 1938.
H. Jeffreys, *Theory of Probability*, 2nd edn., section 5.45, p. 258, Oxford, 1948.

3. *Notation.*—For technical astronomical terms, authors should conform closely to the recommendations of Commission 3 of the International Astronomical Union (*Trans. I.A.U.*, Vol. VI, p. 345, 1938). Council has decided to adopt the I.A.U. 3-letter abbreviations for constellations where contraction is desirable (Vol. IV, p. 221, 1932). In general matters, authors should follow the recommendations in *Symbols, Signs and Abbreviations* (London : Royal Society, 1951) except where these conflict with I.A.U. practice.

4. *Diagrams.*—**These should be designed to appear upright on the page, drawn about twice the size required in print and prepared for direct photographic reproduction except for the lettering, which should be inserted in pencil. Legends should be given in the manuscript indicating where in the text the figure should appear.** Blocks are retained by the Society for 10 years; unless the author requires them before the end of this period they are then destroyed.

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6. *Proofs.*—Costs of alterations exceeding 5 per cent of composition must be borne by the author. Fellows are warned that such costs have risen sharply in recent years, and it is in their own and the Society's interests to seek the maximum conciseness and simplification of symbols and equations consistent with clarity.

7. *Revised Manuscripts.*—When papers are submitted in revised form it is especially requested that they be accompanied by the original MSS.

Reading of Papers at Meetings

8. When submitting papers authors are requested to indicate whether they will be willing and able to read the paper at the next or some subsequent meeting, and approximately how long they would like to be allotted for speaking.

9. Postcards giving the programme of each meeting are issued some days before the meeting concerned. Fellows wishing to receive such cards whether for Ordinary Meetings or for the Geophysical Discussions or both should notify the Assistant Secretary.

MONTHLY NOTICES

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ROYAL ASTRONOMICAL SOCIETY

Vol. 115 No. 1

MEETING OF 1955 JANUARY 14

Dr J. Jackson, President, in the Chair

The election by the Council of the following Fellows was duly confirmed :—

Cedric Arnold Bell, B.A., Air Commodore, Castle Wood House, New Oxford Street, London, W.C.2 (proposed by A. C. B. Lovell);

Mary Bradburn, M.Sc., Ph.D., Royal Holloway College, Englefield Green, Surrey (proposed by W. H. McCrea) ;

John Eric Bullard, C.B., Christ's Hospital, Horsham, Sussex (proposed by G. J. Whitrow) ;

John Corner, M.A., Ph.D., Atomic Weapons Research Establishment, Aldermaston, Berkshire (proposed by D. H. Sadler) ;

*Alan Arthur Day, B.Sc., Trinity College, Cambridge (proposed by B. C. Browne);

Tao Kiang, B.Sc., University of London Observatory, Mill Hill Park, London, N.W.7 (proposed by C. W. Allen);

Frank William Land, M.Sc., Ph.D., 129 Beresford Road, Oxton, Birkenhead, Cheshire (proposed by J. Kershaw);

*Alan Pennell Lenham, 43 Newcastle Street, Swindon, Wiltshire (proposed by P. Moore);

*Brian Miles, 14 Sandacre Road, Baguley, Wythenshawe, Manchester (proposed by Z. Kopal);

*Bernard Ephraim Julius Pagel, Sidney Sussex College, Cambridge (proposed by G. Burbidge);

*George Alan Wilkins, B.Sc., A.R.C.S., D.I.C., Ph.D., H.M. Nautical Almanac Office, Herstmonceux Castle, Hailsham, Sussex (proposed by D. H. Sadler); and

*Derek Henry Wilson, M.Sc., University College, Achimota, Gold Coast, West Africa (proposed by H. Bondi).

One hundred and two presents were announced as having been received since the last Meeting, including :—

Sir William Herschel, *Manuscript letter to Sir Joseph Banks, describing his discovery of the new planet Uranus, and dated 1781 November 19* (presented by Sir Eric Miller).

* Transferred from Junior Membership.

The President announced that the Council had awarded the Gold Medal of the Society to Professor Dirk Brouwer, Director of the Yale University Observatory, for his outstanding contributions to celestial mechanics.

The President announced that the Council had awarded the Eddington Medal to Professor H. C. van de Hulst, of the Leiden Observatory, for his prediction of the 21-centimetre line radiation of neutral hydrogen and for his share in its detection and observation, and in the theoretical interpretation of the observations.

ON THE FORMATION OF GALAXIES IN A STEADY STATE UNIVERSE

D. W. Sciama

(Received 1955 February 12)*

Summary

In this paper we propose a mechanism for the formation of galaxies which depends on a characteristic feature of the steady state universe, namely, that any particular galaxy is formed in a universe already full of galaxies. The gravitational fields of these pre-existing galaxies produce density concentrations in the intergalactic gas so that new galaxies can form as a result of gravitational instability. The condition that the properties of this system of galaxies be independent of time suffices to determine these properties uniquely, i.e. without any adjustable parameters. This theoretical distribution is in reasonable agreement with observation for the average mass, size, separation, peculiar velocity and clustering tendencies of galaxies.

These results suggest that the steady state theory may enable one to account for the *actual* distribution of matter in the universe entirely in terms of the *general* laws and constants of nature.

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1. *Introduction.*—The formation of galaxies is a process of astrophysical interest both for its own sake and because it determines the initial conditions for the evolution of many astrophysical systems. In addition, the process is of cosmological interest since the distances and times involved are so large that the differences between rival cosmological theories are important. There have been attempts to discuss the formation of galaxies without the use of a detailed model of the universe (e.g. Weizsäcker (**1**), Hoyle (**2**)), but then certain *ad hoc* assumptions had to be made. To avoid having to make such assumptions we shall work in direct contact with the cosmological problem.

The formation of galaxies in model universes which evolve from a highly condensed initial state has been considered by Lemaître and Gamow (**14**–**22**). These authors studied the possibility that local density concentrations would arise which would be stabilized by self-gravitation. In Gamow's most recent work (**16**) these concentrations are considered to be a result of the highly supersonic turbulence which is set up in the expanding gas because of its enormous Reynolds number (of order 10^6) and high compressibility. These density concentrations will not re-expand if gravitational forces are more important than those arising from pressure gradients. The condition for this to be so has been given by Chandrasekhar (**41**). Gamow uses this condition to estimate the average mass of a galaxy. Taking for the temperature the value derived by the α–β–γ theory of heavy element formation, Gamow obtained the observed mass only if the Mach number of the turbulence is assumed to be of the order of 10. However, the Chandrasekhar theory does not hold for such large Mach numbers (**42**), so that it remains an open question whether or not gravitational condensations can form in systems of the type contemplated by Gamow.

* Received in original form 1954 July 16.

In addition, the Gamow theory contains at least two parameters (the temperature and the Mach number of the turbulence), which are not determined by the basic laws of physics but can be adjusted so as to give agreement with observation. Thus the predictions of the theory are not very specific. Furthermore, this arbitrariness implies that many of the actual properties of the universe are accidental. It would seem preferable to have a theory in which the actual distribution of matter in the universe could be accounted for entirely in terms of the general laws and constants of nature.

We shall attempt to avoid these difficulties by using the steady state model of Bondi, Gold and Hoyle (**3–9**), in which the continual creation of matter enables the large-scale structure of the universe to be independent of time despite the expansion. In this model we are not concerned with the formation of galaxies in an initially homogeneous system, for any particular galaxy is formed in a universe already full of galaxies. These pre-existing galaxies perturb the intergalactic gas in such a way that the density in certain regions is systematically increased, so that new galaxies are readily formed. No adjustable parameters are involved, since the state of the system is uniquely determined by the condition that it be self-propagating and stable. This condition enables one to calculate the average properties of the system of galaxies that exists in a steady state universe.

The properties of the observed system of galaxies for which a theory of galaxy formation can reasonably be expected to account are the following:—

(i) The average mass of a galaxy is about 10^{44} grams (**27**). In addition, there is some evidence (**27**) of a sharp division between heavy-weight and light-weight galaxies with masses of 10^{43} grams and 3×10^{44} grams, respectively. The estimated observational error is less than ± 40 per cent. The figures given here are for double galaxies, but we shall assume that they are universally applicable, since double galaxies have the same dimensions and absolute magnitudes as other galaxies (**30**). However, one must bear in mind the possibility that there may be large numbers of faint dwarf galaxies like Sculptor and Fornax, with masses substantially less than 10^{44} grams.

(ii) The average linear dimension of a galaxy is about 10^{22} cm. This value is somewhat uncertain, since the boundary of a galaxy is difficult to define observationally, and many galaxies possess haloes which extend at least out to distances where their brightness is only one per cent of the background skylight.

(iii) The average period of rotation of a galaxy is about 10^{8} years (**39, 40**).

(iv) The average peculiar velocity of a galaxy (deviation from the Hubble velocity) is about 10^{7} cm/sec (**39**).

(v) The average distance between galaxies is about 10^{24} cm (**39**).

(vi) The galaxies show a strong clustering tendency. It is extremely difficult to characterize this tendency in detail (**12, 31**), but there is no doubt that there are identifiable clusters of a thousand or more galaxies, as well as smaller clusters. A recent analysis (**12**) of Shane's galaxy counts suggests that there are very few galaxies which do not belong to a cluster, but unfortunately this appears to be the only reliable information that emerges from the analysis.

We shall see that our theory is able to account roughly for all these facts. In addition, detailed predictions can be made about the statistics of clusters. We begin by considering the formation of single galaxies in the neighbourhood of single galaxies. Clusters are considered in Section 3.

2. *The formation of single galaxies*

2.1. Consider a single galaxy of mass M_1, moving relative to the intergalactic gas in its neighbourhood with velocity V. The origin and value of this "peculiar" velocity will be discussed later (Section 3.3). Relative to the galaxy the intergalactic gas is streaming past with velocity V, and is being perturbed by the gravitational action of the galaxy, as in the accretion theory of Hoyle, Lyttleton and Bondi (**23, 24, 25**). As a result there is an increase of density in the wake of the galaxy which enables a new galaxy to form (*cf.* **26**). We now discuss the details of this process.

An exact discussion of the dynamics of the process would require a complete relativistic treatment, but fortunately relativistic refinements are not important. It can be shown (**10**) that for our purpose a sufficiently good approximation consists of assuming that the system is completely Newtonian out to a distance from the galaxy of

$$r_0 \simeq \frac{1}{2}\left(\frac{3M_1}{8\pi\rho}\right)^{1/3},$$

where ρ is the mean density of matter in the universe, and that beyond this distance the galaxy exerts no gravitational effects. If we take Hubble's constant τ to be 5×10^9 years, then ρ is 10^{-28} g/cm³ (since in a steady state universe $(8\pi/3)G\rho\tau^2=1$ (**4, 7, 10**)), and

$$r_0 \simeq 5\times10^8 M_1^{1/3}$$

using c.g.s. units.

Neglecting collisions between gas particles, the equation of motion of a particle is

$$\frac{a^2V^2}{GM_1r} = 1+\cos\theta+\frac{aV^2}{GM_1}\sin\theta,$$

where a is the distance of the particle from the axis before it is perturbed, r is the distance of the particle from the galaxy and θ is its angular coordinate relative to the axis ($\theta=0$ when the particle is on the axis).

Bondi and Hoyle (**25**) had to take collisions into account because they were concerned with the steady state in which, in the absence of collisions, they would be dealing with double-stream regions of dimensions much larger than the mean free path. Here we are only concerned with what occurs before the double stream is set up.

The wake is confined within the hyperbola for which $a=r_0$. This hyperbola cuts the axis at a distance $r_0^2V^2/(2GM_1)$ from the galaxy. This distance turns out to be of order r_0 so that we can consider the wake to lie entirely within the Newtonian region around M_1. The density ρ_p at any point p in the wake is given approximately by

$$\rho_p \simeq \frac{a}{r_p\sin\theta_p}\,\frac{1}{2-(r_p\sin\theta_p/a)}\cdot\rho.$$

This does not vary much over most of the wake, so we shall suppose that the density is uniform and equal to ρ.

The main part of the wake will be unstable against gravitational condensation if its gravitational energy exceeds twice its thermal energy, a criterion first proved by Jeans (**13**). This leads to the condition

$$M_2 \gtrsim 2\times10^{37}T^{3/2}, \tag{1}$$

where M_2 is the mass in grams of the new condensation and T is its initial temperature in degrees absolute. We can estimate this temperature by

considering the work done on the intergalactic gas by the galaxy M_1. This work produces an energy change of order $GM_1/(10r_0)$, since a typical particle will change its distance from M_1 by a quantity of order $(\frac{1}{10})r_0$. If we assume that the gas is hydrogen, this leads to a temperature T given by

$$T \simeq 5 \times 10^{-17} M_1/r_0$$
$$\simeq 10^{-25} M_1^{2/3}.$$

If we insert this in (1) we obtain

$$M_2 \geqslant 0{\cdot}5 M_1.$$

The lower limit corresponds to the case where a mass only just forms in an infinite time, so the practical lower limit will be somewhat larger than this, say M_1, for which the formation time is of order $(G\rho)^{-1/2} \simeq \tau$. This lower limit will only be realized if there is sufficient material available in the wake. In order to estimate the mass available we shall suppose that the effective wake is half the actual wake, so as to allow for the fact that the galaxy M_1 will control the material in its vicinity. The total available mass is then

$$\frac{\pi r_0^4 V^2}{4GM_1}\rho.$$

This equals our theoretical lower limit for the mass of M_2 if

$$V \simeq 10^{-7} M_1^{1/3}. \tag{2}$$

We shall see later that this relation is satisfied by a small margin. If in a particular case the velocity V is considerably greater than the value given by (2), the mass available would permit two galaxies to form, but for the moment we shall ignore this possibility.

2.2. The work of Section 2.1 has shown that so far as the Jeans criterion is concerned, and neglecting any possible change of mass with age, galaxies of any mass M will give rise to galaxies of the same mass M, so that we cannot in this way determine the average mass of a galaxy. The reason for this is that the only basic quantities we have used are the gravitational constant and the mean density of matter in the universe. We cannot build from these a quantity of the dimensions of mass, so that we are dealing with a set of dynamically similar systems. In order to be able to determine a mass we need in addition a quantity of the dimensions of length. We can bring a length into the theory by imposing the condition that the condensing galaxy must radiate an appreciable amount of energy. This condition is needed to ensure permanent contraction, since if there were no loss of energy the system would re-expand to its original size. This condition involves the thermal properties of hydrogen, which depend on a quantum-specified length, e.g. the Bohr radius.*

The relevant thermal properties of hydrogen have been discussed by Hoyle (2). The results of interest to us represent an interplay between three factors:—

(i) The internal energy of hydrogen.

(ii) The energy converted into radiation by means of free-free and free-bound electronic transitions. Most of this energy escapes from the system, thereby lowering its total energy.

(iii) The energy converted into heat as a result of gravitational condensation.

* I am very grateful to Mr F. Hoyle for pointing out to me the relevance of the thermal properties of hydrogen, and for showing me his paper (2) before publication.

Hoyle's work shows that for an initial density of about 10^{-28} g/cm^3 the radiation condition can only be satisfied if the kinetic temperature T is either about 10^4 deg. K or about 10^5 deg. K. The reasons for this remarkably simple result are the following. Below 10^4 deg. K the number of free electrons is very small, so that there is no appreciable radiation. At 10^4 deg. K hydrogen begins to be ionized by collisions and radiates appreciably. For temperatures in the neighbourhood of 10^4 deg. K this radiation acts as a thermostatic control during gravitational condensation. For if the temperature were to drop below 10^4 deg. K, radiation would cease and the temperature would rise. On the other hand, if the temperature rose much above 10^4 deg. K, cooling by radiation would exceed the heating due to compression and the temperature would fall again. Thus in this region of temperature, gravitational condensation is an isothermal process.

Between 2×10^4 deg. K and 10^5 deg. K the rate of radiation decreases again to such an extent that the energy input required to produce a temperature of 10^5 deg. K is only slightly greater than that required to produce 2×10^4 deg. K. Thus intermediate temperatures are unlikely. At 10^5 deg. K cooling by radiation is still sufficient for contraction to be an isothermal process, but this is no longer true for temperatures much in excess of this. Hence, as stated above, the radiation condition is only satisfied for $T\simeq10^4$ deg. K or $T\simeq10^5$ deg. K. As Hoyle pointed out, this may be the explanation of Page's (**27**) discovery that there appears to be a sharp division between light-weight and heavy-weight galaxies, with only a few intermediate ones.

Let us first consider the low-temperature case. Putting $T\simeq10^4$ deg. K in (1) yields

$$M\geqslant 2\times10^{43}\,\mathrm{g}.$$

This is in reasonable agreement with Page's value of 10^{43} g for the light-weight galaxies. With this value of the mass, equation (2) for the peculiar velocity V becomes

$$V\simeq 2\times10^{7}\,\mathrm{cm/sec}.$$

We shall see in Section 3.3 that this condition is satisfied theoretically. It is also satisfied by observation since peculiar velocities are of the order of 10^7 cm/sec (**8**).

For the high-temperature case we have

$$M\geqslant 6\times10^{44}\,\mathrm{g},$$

which may be compared with Page's value of 3×10^{44} g for the heavy-weight galaxies. The peculiar velocity now satisfies

$$V\simeq 6\times10^{7}\,\mathrm{cm/sec},$$

which is still reasonable for many galaxies.

2.3. So far we have considered the conditions that must be satisfied for a new galaxy to begin to form in the neighbourhood of a fully-formed galaxy. We must now examine the subsequent development of this system. The relative velocity of the two galaxies will be of order V, which is of the same order as the escape velocity. Hence the child galaxy may or may not be captured by its parent, depending on the detailed circumstances. Here we shall suppose that it escapes, the case of capture being discussed in Section 3.1. We thus end up with two single galaxies each with a velocity of about $\frac{1}{2}V$. This damping of the peculiar velocity does not imply that in the steady state single galaxies have no peculiar velocity; this is discussed in Section 3.3.

The development of the contracting galaxy presumably follows the lines laid down by Hoyle (**2**), who showed that a hierarchical structure will arise within the galaxy. Subcondensations form as soon as the galaxy contracts a little. Then further subcondensations form within these and so on, in a time much less than that required for the first condensation. After about 13 steps the densities in the final subcondensations are so great that radiation can no longer escape, and the isothermal approximation must be replaced by an adiabatic one. The fragmentation process now ceases. It is very satisfactory that the final subcondensations have masses of stellar order, and can be identified with type II stars. These considerations show that if the galaxy has no angular momentum its linear dimensions shrink by a factor of about 40 before contraction ceases (the details are given in Hoyle's paper). For an initial density of 10^{-28} g/cm^3 this implies a final density of about 6×10^{-24} g/cm^3, in reasonable agreement with observation. Those galaxies which do rotate will of course contract more in the polar than in the equatorial direction. Indeed, in spiral galaxies the configuration of the disk is almost entirely determined by centrifugal forces (**28**). We proceed to calculate their value.

Following Hoyle (**29**), we ascribe the origin of the rotation to the gravitational couple exerted by surrounding galaxies on the irregularly shaped initial configuration of the condensing galaxy. By symmetry, the parent galaxy has only a small effect, so that the neighbouring galaxies must be responsible for the couple. This couple leads to a final angular velocity, after contraction has ceased, of (**29**)

$$\Omega \simeq 10^{-98}\, d^9/M^3x^3,$$

where M is the effective mass of the perturbing field, d is its effective distance from the centre of the condensing galaxy and x is the largest of the three quantities $(B-C)mn/A$, $(C-A)nl/B$ and $(A-B)lm/C$. Here A, B, C are the principal moments of inertia of the condensing galaxy in the early stages of condensation, and l, m, n are the direction-cosines of the line joining the centre of the galaxy to the centre of gravity of the perturbing field. In order to compute Ω, we shall take $d\simeq10^{24}$ cm (see Section 2.4), $M\simeq10^{44}$ g and $x\simeq0{\cdot}25$. This estimate of x follows from the fact that the condensing galaxy is initially considerably elongated, so that the maximum value of $(A-B)/C$ is of order unity. x is then just the product of two direction-cosines, and will in general be of the order of a few tenths. We thus have

$$\Omega \simeq 6\times10^{-15}\ \mathrm{sec}^{-1}.$$

This is of the right order of magnitude, but the result depends so critically on M, x and particularly d, that the agreement is not very significant.

2.4. So far we have only considered a single birth-process. In order that the indefinite repetition of this process should lead to a steady state distribution of galaxies, its time-scale must have a certain average value, t_0 say (of order τ). Otherwise, galaxies will not reproduce at the rate required to keep their number-density constant despite their mutual recession. We have seen that the time-scale for a birth-process is of order τ, and so it is of order t_0. It must not exceed t_0, for then the number of galaxies per unit volume would decrease indefinitely with time, and the steady state number-density of galaxies would be zero. We cannot expect the time-scale to be exactly equal to t_0, so we must assume that it

is less than t_0.* This does not imply an indefinite increase in the number-density of galaxies, since the finite rate of creation of matter places an upper limit on the amount of material available for forming galaxies. The number-density of galaxies will be the maximum consistent with this restriction. Now enough material is created within a sphere of radius r_0 in a time τ to form a mass of order M. Hence the mean separation of galaxies will be about $3r_0$, allowing for the recession that occurs before new galaxies can form. For the minimum values of M deduced above the mean separation is about 4 to 10×10^{23} cm, in reasonable agreement with observation (**8**).

2.5. We have now shown that, neglecting clustering which we consider in the next section, the self-propagating system of galaxies that can exist in a steady state universe has the same general properties as the observed system of galaxies. However, before we can identify the two systems, we must be able to show that the theoretical one is stable and is, indeed, the only stable one. The stability follows readily from the thermal properties of ionized hydrogen. For if by chance a galaxy is lighter than average, then the initial temperature of its child will be lower than average. But since the amount of cooling is less at lower temperatures, this drop in temperature will be less than corresponds to dynamical similarity. A similar argument applies to galaxies heavier than average. Hence the masses of the children will be nearer to the average than those of their parents. Since the properties of the system are determined by the average mass of a galaxy, this means that the system is stable.

It remains to show that no other self-propagating distribution is stable. The only other such distribution is the homogeneous one, which conforms precisely to the de Sitter metric. It is known that this system is stable against infinitesimal fluctuations, but it follows from the work of this section that it is not stable against finite ones. Thus if a fluctuation leading to a galaxy of mass about 10^{43} g occurs, then eventually the universe will be filled with such galaxies. Of course such a fluctuation will take a very long time to occur, but there is infinite time available. We conclude that our self-propagating distribution of galaxies must arise in a steady state universe of infinite age.

3. *The formation of clusters*

3.1. In the previous section we supposed that the child galaxy escaped from its parent. Since the recession velocity is of the same order as the escape velocity, in about half of the cases the child will be captured. In this section we suppose this to occur, and study the subsequent development of the system.

When capture occurs a double galaxy is formed. In most cases the galaxies will not collide, as neighbouring galaxies will perturb the child into a bound orbit. The size of this orbit can be estimated from the theory of the last section—it is about $(\frac{1}{4})r_0$. For the light-weights this corresponds to a mean radius of 5×10^{22} cm, in reasonable agreement with observation (**30**).

This double galaxy can now act as a parent in another birth-process. The peculiar velocity of its centre of gravity will be about half that of the original galaxy, so that it is almost certain to capture its children. The details of the process are similar to those discussed in the last section, since in the relevant regions of space the double galaxy acts gravitatively as a single mass at its centre of gravity. Since the mass of the perturbed gas cloud is of the same order as

* We have to make an assumption at this point because although the time-scale is certainly of order t_0, we cannot hope to calculate it precisely enough to prove that it is less than t_0.

the mass of the double galaxy, the cloud will split up into one or two new galaxies, depending on the detailed circumstances. In this way a cluster of three or four galaxies is formed.

The peculiar velocity of this cluster is only about a quarter of the original peculiar velocity, so that at about this stage of development the system can be regarded as at rest relative to the intergalactic gas. The parent now acts spherically symmetrically on the intergalactic gas, so that the new galaxies form in a shell around the parent cluster. If account is taken of the matter created in the condensing gas (**10**), there is enough material available to double the number of galaxies in the cluster at each birth-process. That the cloud does split up into galaxies of the same mass as those in the parent cluster follows from the thermal considerations of the last section. For large clusters the temperature is more likely to be in the region of 10^5 deg. K than 10^4 deg. K (since $T \simeq M^{2/3}$), so that we should expect large clusters to contain a relatively greater number of heavy-weights. A further possibility (**2**) is that as contraction proceeds the rate of radiation becomes so great that the temperature falls abruptly to about 10^4 deg. K. At this stage the condensing gas has a density considerably greater than 10^{-28} g/cm^3, so that the minimum mass that can form will be less than our previously calculated value. Hoyle suggests that this process may account for the dwarf galaxies that are now being observed in increasing numbers.

Very massive clusters will induce in the surrounding gas a temperature far in excess of 10^5 deg. K, despite the cooling by radiation, so that this gas will not condense into new galaxies. Hence there is an upper limit n_0 to the number of galaxies in a cluster. A rough estimate based on these thermal considerations gives for n_0 a value lying between 10^3 and 10^4. The results of this paper which depend on n_0 do so only logarithmically, so that we do not need a more precise estimate.

We have here a mechanism for the formation of clusters of all sizes up to about 10^4 galaxies. In order that a steady state distribution be set up a time-scale condition must be satisfied just as for single galaxies. Thus each birth-process must take rather less than $t_0 (\simeq \tau)$, so that the limit is set by the availability of material. As before, all we can say is that the birth-time is of order τ, and we have to assume that it is actually less than t_0.

A further condition must be satisfied before a steady state distribution of clusters will be set up. For our mechanism implies that at each birth-process about half the single galaxies become double galaxies. Hence we need a source of single galaxies to replace the ones consumed. If there were no such source, there would be no galaxies at all in the steady state. Fortunately a sufficient source is provided by the dynamical evaporation of single galaxies from clusters. This process arises because from time to time single galaxies in clusters obtain enough energy from gravitational interactions to enable them to escape from the cluster. The cluster then settles down into a more compact state until the process occurs again. We shall show later that this process is sufficient to supply the singles needed. Here we shall suppose that this is so, and proceed to calculate the details of the distribution of clusters.

3.2. We shall begin by assuming that every single galaxy captures its child; this assumption will be dropped later. Since in the steady state the contents of a cluster increase at the same rate as a freely expanding volume, the number n of galaxies in a cluster of age a (neglecting evaporation) is given by

$$n = e^{3a/\tau}, \tag{3}$$

where the zero of age corresponds to one galaxy. The age a_0 corresponding to the maximum size n_0 is thus about 3τ. The number of clusters per unit volume of age between a and $a+da$ is (6)

$$(3\alpha/\tau)e^{-3a/\tau}\,da, \tag{4}$$

where α is the total number of clusters per unit volume. Hence the number of clusters containing from $n(<n_0)$ to $n+dn$ galaxies is

$$\alpha\, dn/n^2$$

and the number containing n_0 galaxies is α/n_0. These predictions cannot as yet be compared with observation, since even the most recent galaxy counts (31) do not lead to reliable results. The total number of galaxies per unit volume is

$$(3\alpha/\tau)\int_0^{a_0} da + \alpha \simeq 10\alpha. \tag{5}$$

It follows that the average number of galaxies per cluster is 10, which seems a reasonable result.

To determine α, we identify (5) with the total number of galaxies per unit volume permitted by the availability of material; this was discussed in Section 2.4. This gives for the total number of clusters per unit volume

$$\alpha \simeq 2\times 10^{-74}.$$

Thus in a sphere of radius 10^9 light years (corresponding to a limiting magnitude of about 18) there will be 100000 clusters, containing altogether a million galaxies.

Since a galaxy remains single for a time of order $\tau/4$, the number of singles at any instant is

$$(3\alpha/\tau)\int_0^{\tau/4} e^{-3a/\tau}\,da = \alpha/2.$$

Hence half the "clusters" consist of single galaxies, and about 5 per cent of all galaxies are single. The number of apparently single galaxies will be somewhat greater than this (see later). In addition, if only, say, half the singles capture their first child, then the proportion of singles is nearly doubled and the number of clusters is slightly reduced. Thus we should expect about 10 per cent of galaxies to be single. This is a much smaller proportion than was at one time thought, but it is in line with the recent tendency to regard most galaxies as belonging to a cluster.

One of the most noticeable features of galactic clustering is the existence of large clusters containing a thousand or more galaxies. It is therefore important that the theory should be able to account for such large clusters, as well as for the smaller ones such as the local group. Now the number of clusters per unit volume of age exceeding a is

$$\alpha e^{-3a/\tau} \simeq \alpha/n,$$

where n is the number of galaxies in the smallest such cluster. Hence in a sphere of radius 10^9 light years the number of clusters containing a thousand or more galaxies is about 100, and the number containing $n_0(\sim 10^4)$ is about 10. This is a reasonable result.

We now consider the distances between clusters. Since the number of clusters containing n or more galaxies is proportional to $1/n$, their mean separation is proportional to $n^{1/3}$. Hence clusters of a thousand or more galaxies should

be separated by ten times the mean cluster distance, that is by $10\alpha^{-1/3} \simeq 4 \times 10^{25}$ cm or 4×10^7 light years. Two such clusters can never be closer than this, for otherwise they would have captured each other in the past when they were smaller but much closer together. This consideration gives a general restriction on the closest distance between any two clusters. For consider two clusters which at time $t=0$ have ages a and b, $(a<b)$, and are a distance d apart. Then at time $-t$ their distance apart was d_0, where

$$d = d_0 e^{t/\tau}.$$

If we take t such that d_0 is the radius of influence of a galaxy ($\sim 10^{24}$ cm), then at time $-t$ the younger cluster must not have been born, otherwise it would have been captured by the older cluster. Hence the maximum age of the younger cluster is t, and so the number n of galaxies in it must satisfy

$$n < e^{3t/\tau} = (d/d_0)^3.$$

3.3. We thus see that if we assume that the supply of single galaxies by evaporation from clusters is maintained at the required rate, then the resulting steady state distribution of clusters is in reasonable agreement with observation. We now show that this assumption is plausible. In order to do this we must know the radius R of a cluster containing n galaxies. We shall suppose that the mean intergalactic distance in a cluster is the same as that we estimated for a double galaxy, namely 5×10^{22} cm. This is rather a crude assumption, but it suffices for our purpose. In order to test the assumption we write

$$\begin{aligned} R &\simeq 5 \times 10^{22} n^{1/3} \\ &\simeq 10^8 M^{1/3}, \end{aligned} \tag{6}$$

where M is the mass of the cluster, and we have taken 10^{44} g as the mass of a galaxy. Now the mass of a cluster is not directly observable, and the number of galaxies it contains is uncertain as there may be many faint ones. We thus prefer to work in terms of the mean square velocity $\overline{v^2}$ of galaxies in the cluster, which, if the cluster is in a steady state, is related to M and R by (**32**)

$$\overline{v^2} = \tfrac{1}{2} GM/R.$$

We shall see later that there is adequate time for a steady state to be set up in the cluster. Hence we can write

$$(\overline{v^2})^{1/2} \simeq 1{\cdot}7 \times 10^{-16} R,$$

a relation between two quantities that are directly observable. For the Virgo cluster $R \sim 10^{24}$ cm (**33**), so $(\overline{v^2})^{1/2} \simeq 1{\cdot}7 \times 10^8$ cm/sec, in reasonable agreement with the observed value of 10^8 cm/sec (**34**). For the Coma cluster the agreement is not so good: $R \simeq 3 \times 10^{24}$ cm (**35**), so $(\overline{v^2})^{1/2}$ should be 5×10^8 cm/sec, whereas it is only 2×10^8 cm/sec (**35**). This shows that our assumption is not very accurate, but as we shall see, this does not matter for our present purpose.

Before we proceed to calculate the rate of evaporation, it is convenient to describe here the source of the peculiar velocities of single galaxies, which, as we saw in Section 2.3, are reduced at each birth-process. The source is the dynamical interactions within clusters. As we have said, these interactions lead to some galaxies having large velocities. If these velocities exceed twice the root-mean-square velocity, the galaxies escape. The velocities involved are of order 10^8 cm/sec, but these are reduced to a few times 10^7 cm/sec by the time the galaxy has completely escaped. This is just the initial value of the peculiar

velocity used in Section 2.2. We may also note in passing that the lighter galaxies tend to escape preferentially (**36**); this is another reason for expecting the heavier galaxies to be more concentrated in large clusters.

In order to calculate the rate of evaporation, we must know the relaxation time of a cluster—this is the time it takes for an approximately Maxwellian distribution of velocities to be set up from an arbitrary initial distribution. This time is given by (**36, 37, 38**)

$$t \simeq 3{\cdot}5 \times 10^{-6}(nR^3/m)^{1/2}/(\log_{10} n - 0{\cdot}45) \text{ years},$$

where R is the cluster radius in centimetres and m is the average mass of a galaxy in grams. Using (6) we have

$$t \simeq 3{\cdot}5 \times 10^6 n/(\log_{10} n - 0{\cdot}45) \text{ years}.$$

For $n \simeq 10^3$, $t \simeq 10^9$ years, and is correspondingly less for smaller clusters. Thus, except for the largest clusters, there is ample time for a steady state to be set up within a cluster (*cf.* (**38**)).

As Chandrasekhar (**36, 37**) has shown, the number of galaxies that will evaporate in a time $\tau/4$ is

$$\begin{aligned} &0{\cdot}0075 n\tau/4t \\ &\simeq 1{\cdot}2(\ln n - 1). \end{aligned} \qquad (7)$$

This does not depend strongly on n because larger clusters have a longer relaxation time, so that they take longer to regain a Maxwellian distribution after each galaxy escapes. The total number of galaxies lost is

$$N \simeq 1{\cdot}2\alpha \int_6^\infty (\ln n - 1) n^{-2}\, dn,$$

where we have taken for the lower limit $n = 6$, for which (7) gives one galaxy lost—statistical arguments will presumably not apply to a smaller assembly. Hence

$$N \simeq 0{\cdot}4\alpha.$$

This is of the same order as the number of singles consumed every $\tau/4$, namely $\alpha/2$. We now argue in the same way as we did about the rate of formation of galaxies. If N were less than $\alpha/2$, the supply of singles would not be maintained, and in the steady state there would be no galaxies at all. Since we cannot expect N to be just equal to $\alpha/2$, we must suppose that it is greater. We then need a mechanism which cuts the number of singles down to $\alpha/2$. This mechanism is provided by the clusters themselves, since they will capture some of the galaxies which have evaporated. This is quantitatively reasonable since the mean free path of a single galaxy is of the same order as the distance it moves in a time $\tau/4$. The number of galaxies that appear to be single at any instant will thus exceed the number that give birth while single. This point was referred to when we estimated the proportion of single galaxies.

It is not obvious that just the right number of galaxies will be captured so as to leave on the average $\alpha/2$ galaxies to act as parents. However, the following general argument suggests that the distribution of clusters will adjust itself until this is so; at the same time this argument shows that the system is stable. Suppose that by a fluctuation there are fewer clusters than usual in a certain region. Then the number of singles captured in that region will decrease. Thus the number of singles acting as parents will increase, and this will eventually increase the number of clusters. A similar argument applies if the number of clusters initially increases. Hence the system will adjust itself until on the average the

appropriate number of galaxies is captured. This state will then persist indefinitely, since it is stable.

5. *Conclusion.*—Despite the tentative nature of our work, it appears that the steady state theory of the universe provides an adequate framework for a quantitative discussion of the formation, distribution and properties of galaxies, which is free from adjustable parameters and which is in reasonable agreement with observation.

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PROBLEMS OF STARK BROADENING FOR EXPERIMENTAL AND ASTRONOMICAL SPECTRA LIABLE TO SELF-ABSORPTION

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Summary

Experiments are described, relevant to the problem of correlating quantitatively a Stark effect, or any alternative explanation of broadening in spectral lines, with physical state of an atmosphere. A particular difficulty is that of fitting any theoretical contour to an observed line when an unknown degree of self-absorption is suspected of distorting the broadened structure, as in solar flares or radiation from unknown depth in the expanded atmospheres of early-type stars. To study a Stark contour in the presence of self-absorption, the laboratory source chosen is a deep electrodeless plasma with unusually high charge density at low voltage. Contours of Hα are analysed by transmitting from a high-dispersion spectrometer into a photomultiplier through a slit of aperture small compared with line-width. The assumptions required in fitting theories of Stark broadening to contours are examined, to find conditions under which a degree of self-absorption in the core of the line may be estimated. Misfit in the extreme wings is related to some theories which have arisen in stellar physics.

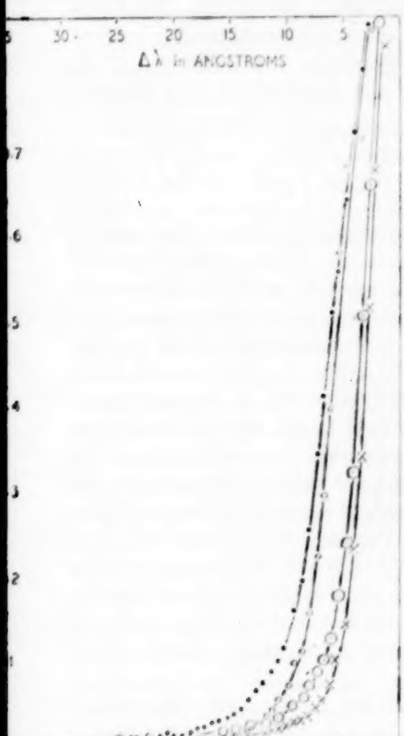
Introduction.—Since the work of Elvey and Struve in 1930 (**1**), a dispersion of optical frequencies by a Stark effect due to charged particles has been widely accepted as one of the major causes of broad wings in astronomical line spectra, particularly of the Balmer series in hydrogen emission and absorption. Some of the difficulties realized by Elvey and Struve, in their recognition that different portions of a single line contour may originate at different optical depth, have not been solved. This leaves ambiguities as to the responsibility for a line structure, for instance in estimating the share of Stark effect or radiation damping or electron scattering or even non-thermal Doppler contributions; the contours for any homogeneous optical depth would not differ greatly among several such sources of broadening. Serious difficulty follows in attempting to infer from line contours any physical features such as electron concentration. Astronomical instances of current interest include the Be atmospheres (**2**) where some form of electronic broadening of lines has to be separated from effects of rotation, B and A atmospheres (**3**) where Stark effects seem to imply unlikely values of gravity, and the Wolf-Rayet atmosphere (**4**) where structures are now suspected to involve more than solely Doppler effect. In solar physics there is the misfit between Stark theory and infra-red lines (**5**), and many problems concerning flares (**6**). Flare stars may well present the questions in aggravated form when their line structures become available. Some of the difficulties in these problems are summarized by Odgers (**7**) and by Underhill (**8**).

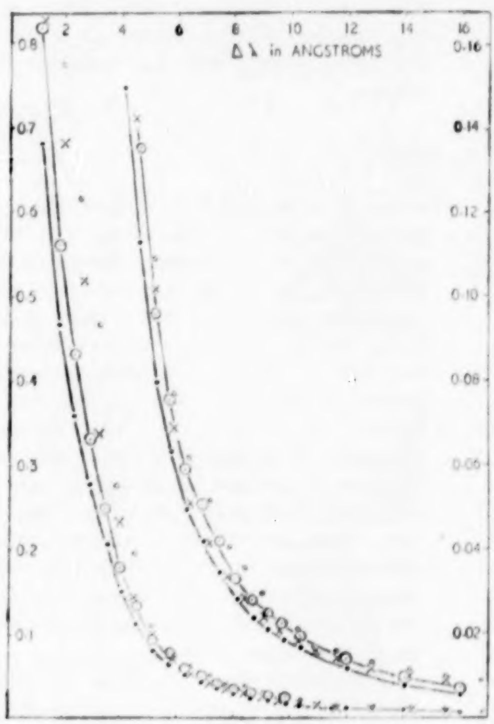
In many of the astronomical examples an unknown quantity seriously vitiating the correlation of observed contours with Stark theory is the degree of self-absorption; the experimental study of Stark broadening, e.g. in small sparks at high voltage, has been more free from this complication than the astronomical where a far larger range of optical depths is usually involved. The purpose of the present laboratory report is to fill a little of the gap between experiment and astronomy, by exhibiting instances where the shares of self-absorption and of Stark effect are both large but are capable to some degree of being distinguished from one another. An unconventional source of Balmer lines is employed, where length of optical path and high concentration of ions at very low voltage can offer a situation more resembling the stellar than the spark sources hitherto common in experimental broadening of spectral lines. Some of the criteria which we obtain for judging stages of agreement between observation and theory may serve in the work of extracting Stark effect from other causes of broadening in astronomical spectra, where distortion by self-absorption can be suspected but not easily proved.

Experimental.—The source of light is the electrodeless ring glow at radio-frequency described in a previous paper on Balmer decrements (**9**). For the present study of line contours the discharge itself is observed, not the afterglow discussed in the earlier paper. The shape of the discharge is a cylindrical plasma about 20 cm long by 6 cm diameter examined from its end; it has longitudinal intensity gradients but little longitudinal speed of particles, the alternating motion being principally circular. Absorbing several hundred watts from a radio transmitter as generator at $3\frac{1}{2}$ Mc/s, the current can be as high as 10 amps, and is here not pulsed but maintained; mass-spectra obtained by the original users of this type of glow indicate that at maximum light the gas shows a proton-electron concentration as high as 97 per cent. This is not so unlike a stellar atmosphere as are most conventional laboratory sources, where speeds are higher but ratio of ion concentration to gas pressure is low; at our usual pressure of 0·1 mm, which is an optimum for high luminosity in resonance with the frequency employed, the concentration of charges can approach its theoretical maximum of 7×10^{15} per cm^3 which would characterize 100 per cent dissociation and ionization and a temperature low compared with stellar. Such concentrations are only commonly exceeded in sparks at much greater pressure.

A multi-prism spectrometer of solar type, with slit of 0·05 mm and a step-wedge, feeds a second slit of 0·05 mm which admits to an 11-stage photomultiplier whose currents are read on a galvanometer sensitive to 10^{-10} amps. By constructing a fine micrometer traverse for the reflector device between the blocks of prisms, actuated by a long arm with cross-wire moving over a large protractor, a broadened Hα can be moved at protractor intervals of $\frac{1}{2}$ degree across the slit of the photomultiplier housing, which accepts about 0·6 A at any one setting. Since we meet a spread of line up to 80 A between wings, and rarely less than 40 A, the resolving power is adequate to study of the wings of a line, and would only introduce instrumental distortion into fine analysis around narrow peaks which is not attempted here. It is quite feasible to obtain 100 photometric points on a contour graph across a single spectral line, with an accuracy exceeding that of most densitometry of photographic plates, and aided by the fact that this type of glow has practically no molecular background.

Some of the data to be used are exhibited in Figs. 1 and 2. Four sample contours of Hα are reduced in Fig. 1 to a common maximum, in the manner familiar for showing progressive self-absorption; they are obtained by slight adjustments in "loading" the oscillator at identical pressure, thus varying considerably the central intensity and the gradients along line of sight near the conditions of resonance between reciprocal frequency and duration of mean free path which permits the very high ionization. Fig. 1 shows the inadequacy of conventional "half-width", and its ambiguous relation to wing-spread and central intensity when Stark effect and self-absorption are superposed; the





1.—*Contours of Hα with common maximum* (= 1·0). FIG. 2.—*Contours of Hα with common maximum* (= 1·0).

● *Theory with no self-absorption.*
○ *Theory with self-absorption.*
× *Observed, rising.*
○ *Observed, falling.*

crossing of wings, when a common maximum is imposed, illustrates the situation discussed by Ellison for solar flares (**6**). A method for distinguishing between Stark effect and self-absorption is attempted below. In Fig. 2 is a single sample, shown both as whole curve and also as 5-fold enlargement of its lower quarter, with the rising and falling sides (× and ○) superposed to indicate the extent of

errors due to wandering instrumental zero and uncertainty in location of exact peak. Theoretical points (● and ○) relate to the errors in assuming that there is no self-absorption, as discussed below.

Methods of comparison with theory.—After the pioneer work of Merton, Hulburt, and others, in proving that spark spectra at atmospheric pressures are mainly broadened by the Stark effect, attention became diverted by astronomers to the detail of extreme wings in stellar lines. Comparison with theory became profitable when Pannekoek and Verwey (**10**) and collaborators obtained on the basis of Holtsmark's assumptions a treatment capable of yielding a synthesis of the Stark components within each line. This treatment provides a graphical distribution of probability among the several field effects within the broadened structure of each Balmer line, in terms of $S(\alpha)$ a statistical function of α, leading to a relation between intensities at variously dispersed wave-lengths. If F_0 is a normal electric field, due to charges each of e and concentrated as N per unit volume,

$$F_0 = 2{\cdot}61eN^{2/3}$$

and

$$\alpha = \Delta\lambda / F_0,$$

where $\Delta\lambda$ is the displacement from the centre of the spectral line. Such graphs are artificial, in that they show zero intensity at the centre of the line and a maximum at some finite value of $\Delta\lambda$, thereafter falling more slowly towards the wings; this is because they take no account of undisplaced components. They have, however, been shown by Craggs and collaborators (**11**) and by Ware (**12**) to allow valuable correlation with the wings of experimental lines: they begin to follow the latter with considerable accuracy at distances from line centre where intensity has fallen to about $\frac{1}{10}$ (Craggs, Hα) or $\frac{1}{8}$ (Craggs, Hβ) or $\frac{1}{12}$ (Ware, Hβ). This correlation permitted an estimation of N reasonably in accord with the circuits used, and the radical divergence when the core of the line is approached was accepted as expressing the inevitable omission of all undisplaced intensities in the theory. Ware noted the very varied stages at which the divergence sets in, ascribing this to the circuit properties which allow greater or lesser luminosity during or outside the instants of greatest ionic concentration. Between wing and core in the astronomical lines, however, control by self-absorption may become dominant in a way which scarcely affects the more shallow sparks.

Accordingly there will be an obstacle to reaching in astronomy the success achieved by those laboratory investigators, in computing a reasonable order of magnitude of N by comparing contours with the theoretical graphs; the comparisons will inevitably become arbitrary when a fit has to be arranged in both coordinates between quantities with no upper or lower limit, especially when it is impossible to know how far across the curve divergence has to be accepted. This suggests for the next stage a quite different method, as follows.

We assume from the degree of success of Craggs and Ware that the theory is capable of a fair first approximation in the wings; we next accept from the mass-spectra already mentioned for our type of discharge that N approaches the ideal 7×10^{15} per cm^3 of a purely proton-electron assemblage. This sets an upper limit not available in the previous experiments, where ions were a small minority in a high gas pressure; the limit could only be exceeded if there is some "pinch" effect of the electromagnetic stresses which will not alter the

order of magnitude.* Calibration for the theoretical intensity distribution to an angstrom scale of abscissae is thereby fixed, since N provides F_0 and thus a value of α for each observed value of $\Delta\lambda$ in the equations quoted.

Calibration of ordinates for testing the Stark theory against measured intensities is then fixed by appeal to another maximum, namely that the intensity in a displaced Stark component cannot *prima facie* exceed that of the undisturbed centre of the line where fields cancel. The peak in the theoretical distribution is therefore allowed first to equal the central intensity of the experimental contour. Fit of theoretical points to experimental curve can then be judged by enquiring whether the agreement improves or degenerates by any adjustment of these two maxima which can be accorded any physical meaning.

The calibrated theoretical points (● ○) are plotted in Fig. 2 against the line selected from Fig. 1 as being the least liable to be affected by self-absorption, i.e. the narrowest in "half-width". The curve which we have thus calibrated is from Unsöld's reproduction (**13**) of Schmaljohann's amended version of Verwey's theory, read by photographing and enlarging under transparent squared paper. Points in the remote wing, where the reproduction becomes inaccurate, are checked by recalculating the theory on Verwey's law of inverse $\frac{5}{2}$ power, which is valid for those large values of $\Delta\lambda$.

Results.—Since any *lower* estimate of N would make any given value of α occur at a smaller $\Delta\lambda$, shrinking the abscissae of the theoretical plot, it appears that an agreement in the wings corresponding to that achieved by the earlier workers is only obtainable by assuming that both ions and electrons are at their highest possible concentration to decide the Stark displacements. We also notice that agreement between theory and observation can extend nearer towards the core of the line if our second restriction as to a maximum were relaxed and the theoretical peak of displaced frequencies admitted to be higher than the experimental maximum. But this would only be permissible as a recognition that the observed peak is one that had suffered depression below some hidden original value by a process of self-absorption in the radiation gradients of the outer plasma—as occurs in a stellar atmosphere but not in a narrow spark.

Accordingly we suggest that the degree of fit between theory and observation may even serve as a useful indicator, in some cases, of where self-absorption is obscuring the contour. The rough agreement of Craggs' and Ware's results in the wings ought to meet no major new obstruction as we approach the core, until the far narrower Doppler width is reached, and indeed if we scale up slightly our theoretical points to fit along the critical region from about $\frac{1}{5}$ maximum down to $\frac{1}{50}$, the similarity of theoretical to experimental track becomes a much closer identity. Such 20 per cent scale-up throughout (points ○ Fig. 2) is equivalent to introducing an assumption that the maximum intensity allowable at the centre, which fixes the scale of ordinates along the curve by setting the height for the theoretical peak, was a pure emission 20 per cent higher than the actual line centre for the same wings, the actual centre having been depressed through self-absorption.

Reverting now to Fig. 1, we may conclude tentatively that the remaining curves are still more strongly self-absorbed in their centres than the one selected

* I owe to a conversation with Dr J. D. Craggs, after writing this paper, the suggestion that N might also be locally increased if plasma oscillations occur and concentrate an excess charge in "ridges" from which the main light comes. Such a phenomenon has sometimes been postulated as a source of solar radio frequencies.

for treatment in Fig. 2, which would accord with their progressive trend in half-widths. To allow any fit of theory to the shoulders of these curves would either (*a*) apply the treatment of Craggs to a misleading domain by allowing a value of N to be computed far exceeding the maximum permitted to such gas pressure, or (*b*) progressively raise the theoretical ordinates as half-widths increase. Our experiments suggest that before a Stark explanation of any broadened line is rejected by failure in the test (*a*), it may often be worth enquiring whether self-absorption has not masked the scale whereby ordinates are fitted, and whether adjustment by the method (*b*) may be feasible.

But although in a simple case a lower limit to the extent of self-absorption is thus obtainable, any upper limit would depend on knowing the quantity which was lacking in the work of Craggs and Ware, i.e. the exact stage at which a theoretical peak may legitimately be accepted as liable to fall short of undisplaced intensities. This would need a precise and unobtainable assessment of all other sources of line broadening beside the Stark effect: comparative treatments for the same physical situation have been rare and incomplete.

Regarding the enlarged section of the shoulder of the curve in Fig. 2, a tendency of the theoretical points to fall below the experimental becomes evident in the extreme wings, although on the scale which includes the core of the line it is scarcely apparent. This sets in at about 14 A from centre, beyond the widths considered in some solar and stellar instances. The effect may be accentuated by the faint background of continuous radiation, since intensities are here only one-hundredth of those in the core; but it is noticeable that Ware's graphs also contain a slight trace of fall in theory at extreme wings, and we refer to some relevant possibilities below.

Astronomical aspects.—The opinion sometimes given in astronomical literature, that calculable Stark effect is inadequate to actual line-widths, may cover two meanings which become separable in our experiment. Firstly, if self-absorption is likely, as in any extended stellar or solar atmosphere, the true central peak of emission at the main site of the radiation is so far in excess of the emergent intensity observable that any theoretical model adequate for the strength in the wings must carry a maximum not limited by that apparent peak. Hence the ordinates throughout any theoretical curve will need scaling-up before it can be judged that it has failed in the wings. It must be noted that our simple scaling-factor was independent of wave-length, relying for its effect on the small alteration in the wings compared with the larger alteration at identical proportionality nearer the core; a second approximation would correct this correction according to dependence of absorption coefficient upon frequency, refining our adjustment if required by the accuracy of measurement. In our experimental plasma, where optical depths are smaller than astronomical though greater than in conventional laboratory sources, it has thus proved possible to estimate the amount of adjustment for self-absorption, in a simple case, by access to a likely value of N. Without such guidance a concentration of charge far exceeding actuality might erroneously be inferred in any attempt to fit theory to the wings. In solar physics, limb-disk comparisons affording a variable range in optical depth may be the corresponding safeguard, and differently orientated rings around binaries may similarly offer the contrast between deep and shallow source in stellar lines.

The second meaning for "inadequacy" of Stark effect is also illustrated in these experiments, if we accept as real the small excess of observed intensity over theory in the extreme wings and do not attribute this entirely to background continuum. The papers of Odgers (**7**) and of Miss Underhill (**8**), drawing attention to the still outstanding problems of extreme wings, recall Eddington's warning that the transit time of fast electrons is not allowed for in the adiabatic theory of Stark broadening; they suggest that fast electrons provide the extreme wings of Balmer lines, although Unsöld had discounted their effect relative to that of ions. We have in the present work included negative as well as positive charges in seeking the value of N which could be large enough, but the introduction of high-speed particles into Stark broadening, by Spitzer and by Krogdahl (**14**), has not yet offered any radical modification of the theories used. In thermal equilibrium high speeds are perhaps not significant, but if the Wolf-Rayet and Be atmospheres and Novae include Stark broadening an allowance for velocity and transit time may be required. In experiments at high voltage this feature might be excessive, accounting for the early divergence of curves in the spark sources, but in our glow at low voltage the speeds of particles will resemble those in an erupting star rather than on the one hand an atmosphere in equilibrium or on the other hand a spark discharge. The discrepancy which begins to show at 14 A in our graph may therefore be relevant to non-thermal astronomical spectra.

Some comparison is finally to be noted with other causes of line broadening. An atmosphere of scattering electrons can present a Balmer contour very like that of a Stark effect. This feature has been most successfully (**4**) invoked for Be and W stars where a very lengthy path is available to obtain the necessary optical depth, and is less likely to affect a laboratory source. The reasons for filling the core of an absorption trough from neighbouring continuum are of less obvious application when fluorescent or flare conditions more remote from thermal equilibrium underlie the luminosity. Burbidge (**2**) has made some progress towards meeting this difficulty, but the demonstration of very large Stark effects in our experiments, at pressures so low that earlier workers would have repudiated the possibility, is a reminder that even in astronomical situations favouring electron scattering the Stark phenomena are not necessarily absent. Radiation damping provides similar contours; in one of the few investigations to cover these simultaneously with Stark broadening, Miss Underhill (**15**) gives reasons why a Stark effect should exceed radiation damping at pressures above a limit exceeded in many hot stars. Since the present work was completed, a paper by Goldberg, Dodson and Müller (**16**) has emphasized the contribution of radiation damping to solar flare contours, as against the Stark explanation by Ellison and Hoyle (**17**). De Jager (**5**) suggested a "saturation" effect after finding Stark contours too narrow to account for solar infra-red hydrogen lines; the saturation phenomenon in equilibrium is the counterpart of what we have regarded as self-absorption in a gas so remote from equilibrium as a laboratory discharge or indeed a solar or stellar flare.

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THE GIACOBINID METEOR STREAM

J. G. Davies and A. C. B. Lovell

(Received 1954 December 8)

Summary

An account is given of the radio echo observations of the return of the meteor shower associated with Comet Giacobini-Zinner during the daytime on 1952 October 9. At this time the Earth was at the node 193 days before the comet. The duration of the shower was only 3 hours, thus showing the same short period characteristics of the great meteoric storms associated with the 1933 and 1946 returns. In 1953 October the Earth was at the node 172 days after the comet but no shower was observed. An interpretation of the factors influencing the occurrence of a meteor shower from the comet is given, and it is suggested that all the results can be explained if perturbations have increased the perihelion distance of the meteors in front of the comet relative to those behind. On this basis a discussion is given of the possibility of observing the shower again in 1958 to 1960. A comment is also made on a recent analysis of the visual, photographic and radio echo observations of the 1946 shower which indicate that a separation of meteor masses may have occurred.

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1. *Introduction.*—The possibility that the debris associated with Comet 1900 III discovered by Giacobini in 1900 December might give rise to a meteor shower was first suggested by Davidson (**1**) in 1915. In 1926 the Earth reached the node 70 days before the comet and an appreciable meteor shower was observed with a radiant near the predicted position. The comet has a period of approximately 6·5 years and subsequently the Earth crossed the orbit 80 days after the comet in 1933 and 15 days after the comet in 1946. The meteor showers observed on these occasions are classed amongst the few great meteoric storms which have occurred during the past century. The failure to observe the shower in the intervening years suggested that the debris might be closely grouped around the comet. On the other hand, the total duration of the shower is only a few hours, and it is possible that in the critical years 1939, 1940, when the Earth was at the node 136 days before and 229 days after the comet respectively, the shower may have been missed because of daylight.

The advent of the radio echo techniques for the study of meteors has enabled the activity of the shower to be assessed without ambiguity since 1946, when it was studied by this technique for the first time (**2, 3**). The present paper describes these systematic observations in the years 1947 to 1954 inclusive, and discusses in detail the unexpected return of the shower during the daytime in 1952, and the significance of its absence in 1953. A full account of the history and the earlier observations of the shower has been given by Lovell (**4**).

2. *Technique and results.*—Observations in each year have been made as part of the continuous meteor survey using the equipment described by Aspinall, Clegg and Hawkins (**5**). Apart from the return of 1952 the observations in each year showed that there was no activity during the Giacobinid epoch in excess of the background sporadic rate (that is, not greater than 4 or 5 per hour). The shower of 1952 was also observed on the original apparatus of Lovell, Banwell

and Clegg (**2**). For the anticipated return of 1953, three additional radio echo equipments were brought into use on wave-lengths between 4 m and 8 m, including a continuously rotating aerial giving coverage of the complete sky. The results were entirely negative.*

3. *The 1952 radio echo observations.*—The radiant survey apparatus (**5**) was in continuous operation during the 1952 epoch, and the range time plots of the two beamed aerials directed at 25° N of W and 25° S of W for the period 1952 October $9^d\,13^h$ to 18^h are shown in Fig. 1(*a*) and 1(*b*). In the equivalent periods on the preceding and succeeding days the average number of echoes was only 4 or 5 per hour. The apparatus with the steerable aerial beam (**2**) was in

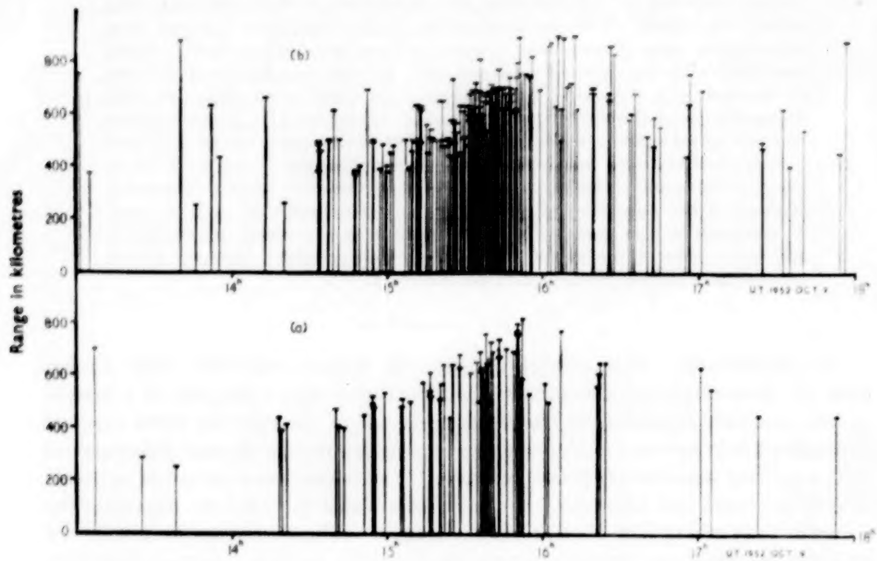

FIG. 1.—*Range-time plots of echoes on the radiant survey apparatus during the Giacobinid epoch 1952 October 9.*

(*a*) *on the aerial directed 25° N of W.*
(*b*) *on the aerial directed 25° S of W.*
Ordinates : Range in kilometres.
Abscissae : U.T. 1952 October 9.

use with the aerial at azimuth 227°, elevation 10° from 1952 October $9^d\,13^h$ to $16^h\,24^m$, and subsequently at azimuth 135°.† The range-time plots are shown in Fig. 2. Unfortunately there is a gap in this record from $14^h\,55^m$ to $15^h\,34^m$ owing to a fault in the recording camera.

(*a*) *Hourly rate.*—The records from which Figs. 1 and 2 were obtained have been analysed to give the rate of echoes on the two equipments. The results are shown in Fig. 3, which gives the number of echoes observed in 10-minute intervals from 14^h to 17^h. The rate plots of the two separate aerials of the radiant equipment were similar and an average has been taken for the plot in Fig. 3. It is clear that activity from the Giacobinid radiant became greater than

* Visual observations in a clear sky by Mr J. P. M. Prentice also gave negative results.

† These aerial positions were those appropriate for the reception of echoes from the Giacobinid radiant.

the sporadic background rate at about $14^h 20^m$. There was a slow rise in activity until 15^h, followed by a rapid increase to a maximum at $15^h 40^m$ on the radiant survey apparatus, and at $15^h 50^m$ on the steerable beam equipment. This difference of 10 minutes in the time of peak activity is probably connected with the slightly different polar diagrams for reception of the two equipments. Unfortunately the radiant was badly placed for observation at this time, being near the zenith at $16^h 25^m$. Under these conditions slight differences in the elevation of the main lobe of the aerial system may influence considerably the

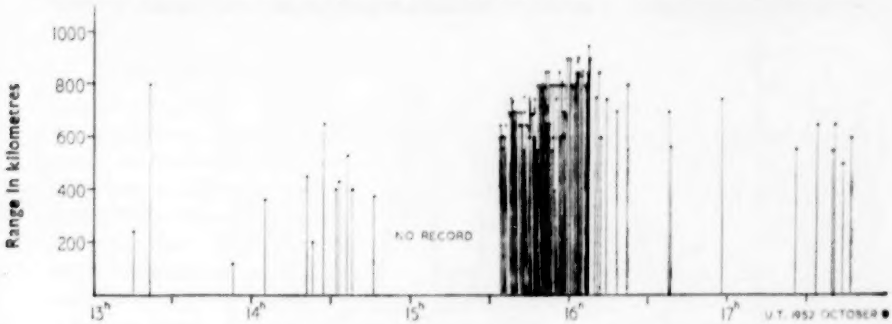

FIG. 2.—*Range-time plots of echoes on the steerable beam equipment during the Giacobinid epoch 1952 October 9.*
Ordinates : Range in kilometres.
Abscissae : U.T. 1952 October 9.

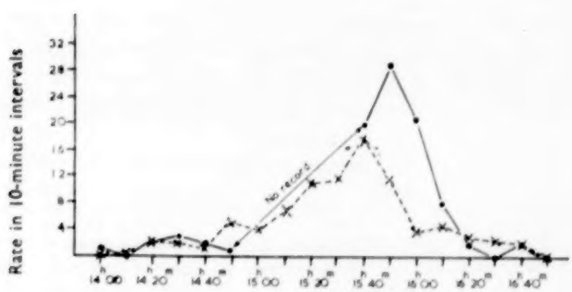

FIG. 3.—*Radio echo rates during the 1952 Giacobinid epoch,*
—×—×— *radiant survey equipment.*
—●—●— *steerable beam equipment.*
Ordinates : Rate in 10-minute intervals.
Abscissae : U.T. 1952 October 9.

recorded echo rates, since the range of the echoes becomes very large from a radiant in the zenith. This same effect may have been responsible for the abrupt fall in rate after $15^h 50^m$ and these records do not exclude the possibility that the actual maximum may have occurred some minutes later.

The question is of some importance in any assessment of the actual hourly rate of the shower at maximum, since in the 1946 results (**2, 4**), the rate increased by at least 5 times in the 10 minutes before maximum. A search for activity after transit proved fruitless, and it is certain that at $16^h 45^m$ there was no activity from the radiant in excess of the sporadic rate. The commencement of the

shower was also well defined on both equipments as illustrated in Fig. 3. It is therefore possible to compare the rate curves for the 1952 shower with those which were well established for the 1946 shower. The appropriate radio echo rate curves for the 1946 shower (2, 4) and the curves of Fig. 3 for the 1952 shower have been normalized and plotted in Fig. 4 against solar longitude. There is very good agreement in the shape of these curves, and both show the same extent in solar longitude. It is, therefore, reasonable to conclude that Fig. 3 represents the actual rate curve of the 1952 shower uninfluenced by the inability of the radio apparatus to record meteors from the radiant near transit.

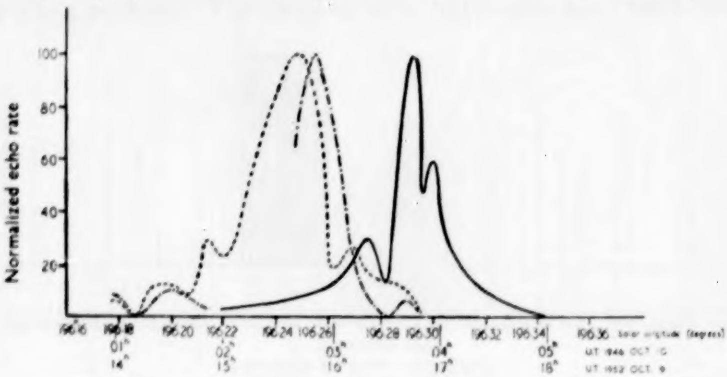

FIG. 4.—*Comparison of the radio echo rate curves for the 1946 and 1952 Giacobinid showers.*

——— 1946.
— — — *Radiant* 1952 } *from Fig.* 3.
—•—•— *Steerable array* 1952 }
Ordinates : *Normalized echo rate.*
Abscissae : *Solar longitude (degrees).*
U.T. 1946 *October* 10; *U.T.* 1952 *October* 9.

TABLE I

Radiant position

	1946		1952	
	Predicted from orbit of comet	Photographic observations	Predicted from orbit of comet	Radio observations
Right ascension	262°	262°·07	262°·2	262°
Declination	+54°	+54°·09	+53°·5	+54°
Solar longitude at maximum	196°·283	196°·297	196°·230	196°·25

The actual hourly rates of echoes on the radiant equipment are standardized by reference to the background sporadic rate (6, 7). In this case the scaling factor is 1·7, which brings the peak of the broken curve in Fig. 3 to the same value as that of the steerable beam curve, giving a rate of 29 echoes in 10 minutes at the maximum or 170 to 180 per hour.

(*b*) ***Radiant position***.—From the range-time plots of Fig. 1(*a*) and (*b*) it is possible to determine the radiant position (5). This is given in Table I

compared with the position predicted from the orbit of the comet. Since the comet has not been recovered the elements used are those given by Dinwoodie (**8**) which allow for perturbations by the Earth, Saturn and Jupiter. For comparison, the data for the 1946 shower obtained photographically by Jacchia, Kopal and Millman (**9**) are included, together with the predictions from the 1946 cometary orbit (**10**).

Both the 1946 and 1952 radiant positions agree with the predicted values within the limits of error of the measurements. The discrepancy in the observed time of maximum from that predicted was $+0^d{\cdot}014$ in 1946 and $+0^d{\cdot}021$ in 1952.

4. *Discussion*

(*a*) *The factors influencing the activity of the meteor shower.*—The intense meteor shower discussed above occurred during the daytime in 1952 when the Earth crossed the cometary orbit 193 days ahead of the comet. Hence, by comparison with the 1926 and 1933 results, there were reasonable grounds for expecting a shower of at least a similar intensity in 1953 when the Earth was at the node 172 days after the comet. As mentioned in Section 2 above, the observations were quite negative. The radiant was well placed for observation on 5 radio echo equipments designed to investigate various aspects of the shower. The predicted maximum occurred during the hours of darkness and visual observations in a clear sky also failed to reveal any activity. It is therefore of interest to consider the various factors which have influenced the intensity of the Giacobinid shower in the past and the reasons for its failure to appear as expected in 1953.

The relevant data since the meteor shower was first observed in 1926 are given in Table II, which shows the distance between the orbits of the Earth and the comet at the node (C–E), the time at which the Earth reached the node relative to the comet, and the hourly rate of the meteor showers observed.

Table II

Year	C–E	Perihelion distance	Earth at node	Hourly rate of meteor shower
	A.U.	A.U.		
1900	−0·0619	0·9319		
1913	−0·0181	0·9759		
1926	+0·0005	0·9947	70 days before comet	17
1933	+0·0054	0·9994	80 days after comet	4000–6000
1939	+0.0012	0·9955	136 days before comet	No shower but predicted maximum occurred in daylight
1940	+0·0012	0·9955	229 days after comet	
1946	+0·0014	0·9957	15 days after comet	4000–6000
1952	−0·0057	0·9887	193 days before comet	180
1953	−0·0057	0·9887	172 days after comet	No shower

It is evident from Table II that neither the distance of the Earth from the node, nor the relative times of passage of the Earth and comet at that point is simply related to the observed hourly rate of the meteor shower. However, if the hourly rate is plotted as a function of both of these quantities as in Fig. 5, the years in which meteor showers have been observed (1926, 1933, 1946 and 1952) all lie close to a curve through the origin, whereas the years with no showers are

randomly distributed at considerable distances from the curve. Thus the three major showers have occurred either when the comet preceded the Earth to the node and C–E was positive, or when the comet followed the Earth to the node and C–E was negative. This suggests that those meteors which reach the longitude of the node before the comet have larger radii vectores at the node than the comet, while those meteors which follow the comet have smaller radii vectores. If this view is correct it is clear that the experimental results give no support to the idea that the debris is concentrated near the comet, and do not exclude the possibility that the debris is distributed all round the orbit.

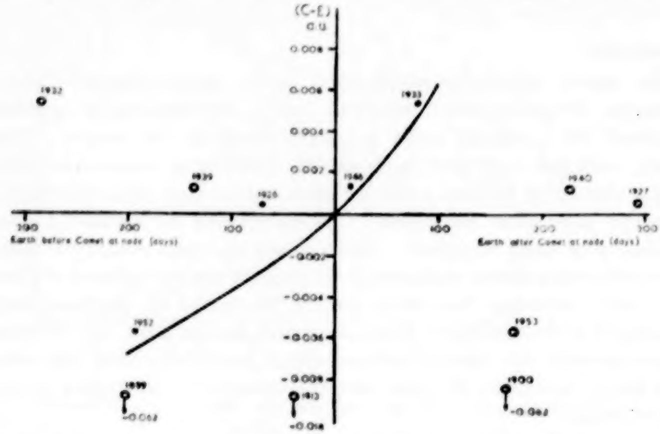

FIG. 5.—*Relation between distance of the Earth from the comet at the node (C–E), and the relative times of passage of the Earth and the comet at that point.*
• *Years of meteor shower.*
o *Years when no meteor shower was observed.*
Ordinates : Distance of Earth from comet (C–E) at node in A.U.
Abscissae : Relative times of passage of Earth and comet at node in days.

(*b*) *The distribution of the Giacobinid meteor debris and the origin of the different radii vectores.*—The observations of the major showers in 1933, 1946 and 1952 are remarkably consistent in the duration. In the three cases the shower was completely contained within 3 hours, while 75 per cent of the peak activity occurred in 1 hour. Hence, all the debris must be moving in the form of a cylindrical stream with a diameter of about 300 000 km, with the dense core 100 000 km in diameter. The 1952 observation shows that this form must be maintained up to at least 5×10^8 km from the comet. It is interesting to notice that although the Earth passed very close to the node of the comet's orbit in 1926 (0·0005 A.U. or 70 000 km) only a minor shower was observed. Reference to Fig. 5 shows, however, that at this point the Earth was 0·003 A.U. or about 400 000 km from the line representing the centre of the concentration along the orbit. As this is 3 stream radii from the line of maximum concentration only a minor shower would be expected.

The effect of perturbations in producing different radii vectores for the particles in such a formation is a complex problem requiring detailed investigation. The comet reaches perihelion close to the descending node and to the Earth's orbit. Aphelion is near the ascending node and the orbit of Jupiter. Since the inclination is 30° the main perturbative force will occur at these two points.

Any solution of this problem on the basis of these forces must take account of the fact that while both period and perihelion distance of the meteors have been altered the diameter of the stream remains very small.

(*c*) *Future occurrences of the Giacobinid meteor shower.*—In 1959 the Earth will cross the cometary orbit 35 days before the comet and on the basis of the 1952 orbit the comet will then lie inside the Earth's orbit at a distance of about 0·0057 A.U. This point lies somewhat further from the line of maximum concentration in Fig. 5 than the 1926 shower, and hence only a weak shower might be expected. In 1960 the orbit will be crossed 330 days after the comet and Fig. 5 indicates that if the comet remains inside the Earth's orbit the debris will not be encountered. On the other hand, the comet approaches Jupiter in 1956–57 and the perturbations will probably result in a further reduction of the perihelion distance. This would reduce the likelihood of even a minor shower in 1959. It is interesting to notice, however, that on the basis of Fig. 5 a perturbation in 1956–57 which caused a significant decrease in the perihelion distance q and hence in the numerical value of (C–E) might bring the 1958 crossing, which will be 400 days before the comet, near the extrapolation of the line of maximum concentration. There is, of course, no evidence as to whether the debris extends to this distance in front of the comet, and the observations in future years will be awaited with great interest.

Finally it may be mentioned that the coordinates of the 1939 and 1940 approaches in Fig. 5 are such that it is very unlikely that any debris was encountered on either occasion. This substantiates the validity of the negative result obtained in these years when the predicted maximum occurred in daylight, and could not have been observed visually in any case.

(*d*) *The separation of meteor masses in the Giacobinid stream.*—In a summary of the observations of the Giacobinid shower Lovell (**4**) has drawn attention to a peculiarity of the activity of the 1946 return. The curve of activity for this shower as measured by the visual (**11**), photographic (**9**) and radio echo techniques (**2**) are compared in Fig. 6. At $3^h 30^m$ all three results show a sudden temporary decrease in activity, and Jacchia, Kopal and Millman (**9**) state that a similar drop was evident in the final frequency curve compiled from all the Canadian visual observations. The radio echo rate reached a maximum at $3^h 43^m$, whereas the maximum in the visual and photographic rates occurred 12 minutes later at $3^h 55^m$, at which time the radio echo rate showed a subsidiary maximum. (The number of observations in the 1952 radio echo results are too small for significance to be attributed to similar subsidiary maxima and minima in the curve of activity shown in Fig. 4.)

These peculiarities in the curve of activity can be explained if a separation of meteor masses has already occurred in the stream. The magnitudes of the photographic meteors (**9**) ranged from 0 to -8. According to the data given by Lovell, Banwell and Clegg (**2**) 80 to 90 per cent of the radio meteors had electron line densities less than 10^{11} electrons/cm, and were therefore infra-visual (Greenhow and Hawkins (**12**), Kaiser (**13**)). Further, the analysis of the durations of the radio echoes (**2**) showed a significant increase in the percentage of echoes of long duration as the shower progressed. Since long enduring radio echoes are correlated with bright meteors (**12, 13**) it seems reasonable to assume that the main peak in the radio echo curve at $3^h 43^m$ was caused by faint meteors, below the photographic and visual limits.

Such a separation of particle sizes could arise from the operation of the Poynting–Robertson effect. According to the calculations of Wyatt and Whipple (**14**) the effect would need to operate for 10^6 years to achieve a separation of meteors of magnitudes $+5$ and -2 in the Giacobinid shower, so that the Earth would pass from one limit to the other in 5 days. In the present observations we are concerned with a similar separation of magnitudes with the Earth passing from one limit to the other in 12 minutes. Interpolation shows that the Poynting–Robertson effect would have to act for 1600 years to achieve this separation.

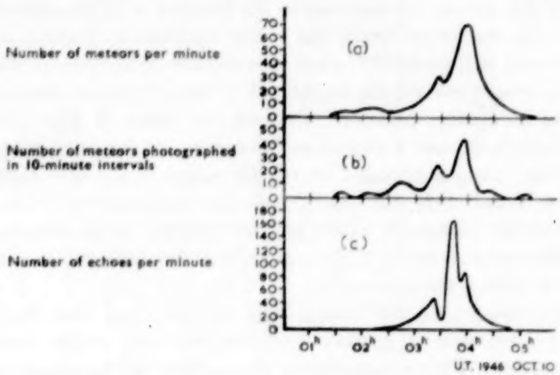

FIG. 6.—*The activity of the Giacobinid meteor shower on 1946 October 10.*
(a) visual, (b) photographic, (c) radio echo.
Ordinates : (a) number of meteors per minute.
(b) number of meteors photographed in 10-minute intervals.
(c) number of echoes per minute.
Abscissae : U.T. 1946 October 10.

Although the comet was discovered as recently as 1900, Fisher (**15**) has concluded that the Giacobinid shower may have been observed as early as A.D. 585. On this basis the time scale would be adequate to achieve the observed separation of particle sizes. There are, however, difficulties in interpreting the effect in this way, primarily because the inclination of the orbit is 30° and the Earth crosses the orbit nearly at right angles to the direction of any separation due to the Poynting–Robertson effect. It is therefore necessary to invoke some other perturbative influence which would change the longitude of the node of the orbits of the smaller particles after separation, differentially with respect to the larger particles.

5. *Acknowledgments.*—The authors are indebted to Mr K. Bullough, who has been responsible for the operation of the radiant survey equipment with which the results shown in Fig. 1 were obtained. They also wish to thank Mr J. P. M. Prentice and many of their colleagues who were present at Jodrell Bank for the 1953 epoch.

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1954 *December* 7.

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THE MASSES OF THE STARS OF POPULATION II

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Summary

Theories of stellar structure and evolution, and the pulsation theory of variable stars, are applied to the determination of the masses of some of the stars of Population II. The most recent data on the colours and magnitudes of the stars are employed in the calculations.

The results, considered with those of other investigators, suggest that the masses of the Population II stars increase from about 0·7 solar masses for the brightest dwarfs in globular clusters, to about 1·0 solar masses for the red giants, variables, blue stars and nuclei of planetary nebulae.

Introduction.—During recent years, evidence has been accumulating which suggests that the masses of the stars of Population II are much less than would be expected from the empirical mass-luminosity relation. For instance, the masses of some sub-giant components of multiple star systems appear to be close to the solar mass, though they are very much brighter than the Sun (**1**). The mass of RR Lyrae appears to be much less than that predicted by the mass-luminosity relation (**2**).

Accurate colour-magnitude arrays of globular clusters (**3, 4**) have made it possible to determine fairly accurately the absolute bolometric magnitudes and surface temperatures, and hence the radii, of some of the stars in them. In the case of variable stars, the application of pulsation theory then enables the masses to be determined, while the problem of determining the masses of non-variable stars may be approached from the point of view of theories of stellar structure and evolution. These are the methods employed in this paper. The results are considered in conjunction with those of other investigators; although there are considerable uncertainties in some of the individual determinations of mass, the overall picture suggests that the masses of the Population II stars increase from about 0·7 solar masses for the brightest dwarfs in globular clusters, to about 1·0 solar masses for the red giants, variables, blue stars and nuclei of planetary nebulae.

The $M_{bol} - \log T_e$ diagram for the globular cluster M3 (**4**) is reproduced in Fig. 1. Included in the diagram is Eggen's sub-dwarf sequence (**5**) (dotted line), which is believed to be the same as the globular cluster main sequences (**6**). The figures and letters on the diagram refer to the notes below it which summarize data on the masses obtained in this paper and by other investigators. The same letters will be used in the text when referring to various parts of the diagram.

Blue stars.—There is a sudden drop in star density at (*i*) on Fig. 1, from the low temperature to the high temperature side (see the original colour-magnitude diagram of M3 (**4**)). This will most probably be because the stars at (*i*) have almost exhausted their immediately available supplies of nuclear fuel and begin to obtain much of their energy from gravitational contraction; they will therefore begin to contract comparatively rapidly, moving quickly across the diagram to the region of very high surface temperatures. Such rapid evolution may therefore be the reason why few stars are observed on the high temperature side of (*i*).

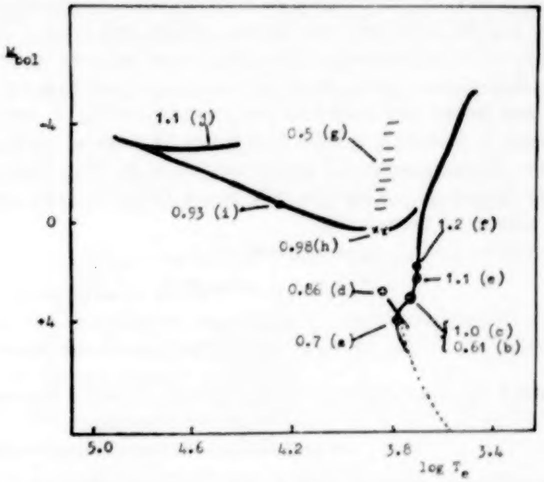

FIG. 1.—*The masses of various types of Population II stars. The figures on the diagram give the masses, while the letters in brackets refer to the type of star or stars and the investigator as follows:*

(*a*) *Brighter sub-dwarfs. This paper.*
(*b*) *Algol 2. Z. Kopal, Ap. J.*, **96**, 399, 1942.
(*c*) *Algol 2. O. J. Eggen, Ap. J.*, **108**, 1, 1948.
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(*f*) *U Sge 2. A. H. Joy, Ap. J.*, **71**, 336, 1930.
(*g*) *Longer period variables in globular clusters. This paper.*
(*h*) *RR Lyrae variables in M92. This paper.*
(*i*) *Blue stars in M3. This paper.*
(*j*) *Nuclei of planetary nebulae; mean of Aller's values. L. H. Aller, Ap. J.*, **108**, 462, 1948.

According to a theory of stellar evolution put forward by the writer (**7**), the brightest red giants in globular clusters evolve to the blue star stage as a result of mixing occurring throughout these stars. Prior to the commencement of mixing, the action of the helium cycle will have produced carbon, oxygen and neon in the cores of the red giants (**8, 9, 10**). These elements will be distributed throughout these stars by the mixing currents. Most of the remaining hydrogen will then be consumed in the carbon-nitrogen cycle, which will also convert much of the carbon into nitrogen.* Thus the stars will reach (*i*) almost devoid of

* The hydrogen will only be completely used up if mixing currents persist after the initial stirring. A recent theoretical investigation by L. Mestel (**11**) of mixing currents due to rotation in Cowling model stars leads to results which suggest that this is unlikely. However, spectral observations do in fact show that in the very blue stars considered in this paper the hydrogen lines have almost, or completely, disappeared (**12**). This is a strong indication that mixing must continue. In view of this apparent conflict of theory and observation in the case of these blue stars, whether or not mixing persists must be considered as yet undecided.

hydrogen, with high abundances of helium, carbon, oxygen, nitrogen and neon, and with nuclear energy being produced by the carbon-nitrogen cycle (**7**). Spectral observations of the stars at (*i*) do in fact show that the hydrogen lines have almost or completely disappeared, and that there are abnormally strong lines of carbon, oxygen and nitrogen (**12**).

Because the helium cycle is highly temperature-dependent, it is probable that the core of a red giant will be convective when the cycle is operating in it (**13**), convection perhaps extending to something of the order of 10 per cent or more of the mass of the star. It may therefore be expected that this proportion of the mass of the star will be converted into carbon, oxygen and neon. The abundance of the Russell mixture in Population II stars is now believed to be only about a tenth of that in Population I stars, that is about 0·2 per cent (**14, 15**). Thus if the red giants become mixed and evolve to the point (*i*) on Fig. 1, the abundance of the oxygen group of elements will be fifty times or more as great as that of the Russell mixture. Consequently the stellar models of M. Hall Harrison, in which elements of the oxygen group are the only heavy elements (**16**), may be used to determine the masses of these stars.

Harrison's values give:

$$\rho_c = 253{\cdot}6\mathcal{M}/R^3, \tag{1}$$

$$T_c = 31{\cdot}9 \times 10^6 \mu\mathcal{M}/R, \tag{2}$$

$$L = \frac{4{\cdot}20\mu^{7\cdot5}\mathcal{M}^{5\cdot5}}{(1+X)ZR^{0\cdot46}}, \tag{3}$$

where $\mathcal{M}$, R and L are in solar units, Z = abundance of heavy elements,
X = hydrogen abundance,
and suffix c denotes central values.

For energy production by the carbon-nitrogen cycle (**10**),

$$\epsilon = 7 \times 10^6 \left(\frac{\rho X}{100}\right)\left(\frac{X_{c+n}}{0{\cdot}01}\right)\left(\frac{T}{3\times 10^7}\right)^{11\cdot0} \text{erg g}^{-1}\text{ s}^{-1} \tag{4}$$

(assuming no resonance in the $N^{14}(p,\gamma)O^{15}$ reaction), where X_{c+n} = abundance of carbon plus nitrogen.

With (**17**)

$$\frac{\epsilon_c}{\bar{\epsilon}} = 38{\cdot}5 \tag{5}$$

equation (4) becomes

$$L_n = 5{\cdot}8 \times 10^7 X X_{c+n} \mu^{14}\mathcal{M}^{16} R^{-17} \tag{6}$$

using (1) and (2), where L_n is the luminosity derived from nuclear energy, in solar units.

If L_g denotes the luminosity derived from gravitational contraction*, and $Q = L_g/L_n$, then the total luminosity is given by

$$L = (1+Q)L_n. \tag{7}$$

Eliminating $\mathcal{M}$ from (3) and (6), and using (7),

$$X(1+X)^{32/11} = \frac{1{\cdot}12\times 10^{-6}}{(1+Q)}\,\frac{\mu^{86/11}R^{172/11}}{X_{c+n}Z^{32/11}L^{21/11}}. \tag{8}$$

* When a large proportion of the luminosity is supplied by gravitational contraction, much of the energy is produced outside the convective core, and hence Harrison's model no longer holds. However, comparison with Levee's gravitationally contracting model (**18**) shows that the difference between the models is small. The mass which will be derived from equation (3) may be about three-hundredths of a solar mass too large.

We also have the well-known formula for the molecular weight,

$$\mu = 1/(2X + \tfrac{3}{4}Y + \tfrac{1}{2}Z), \tag{9}$$

where Y = helium abundance.

For the stars at (*i*) on Fig. 1,

$$M_{\rm bol} = -0{\cdot}9, \quad \log T_e = 4{\cdot}25,$$

giving

$$L = 161\, L_{\odot}, \qquad R = 1{\cdot}33 R_{\odot}.$$

When the helium in the core of the red giant star has all been turned into heavier elements by the action of the helium cycle, $X_{e+n} = \frac{1}{2}Z$ very nearly (**19**). For a first approximation it is assumed that $X = 0$, and μ is obtained from (9) for a chosen value of Z. Then X is obtained from (8) using the value of Q calculated in the manner described below (repeating if X is not very small) and the mass $\mathcal{M}$ follows from (3).

To determine Q.—The gravitational energy of a star* is equal to $\frac{3}{4}\beta G\mathcal{M}^2/R$, where β is the ratio of the gas pressure to the total pressure (**20**). For the masses considered here, $\beta \simeq 1$, and hence the rate of release of gravitational energy is given by

$$L_g = -7{\cdot}6 \times 10^{14}(\mathcal{M}^2/R^2)dR/dt. \tag{10}$$

where $\mathcal{M}$ and R are in solar units.

The rate of consumption of hydrogen is proportional to the nuclear energy production; since the amount of energy released when one gram of hydrogen is converted into helium is $6{\cdot}6 \times 10^{18}$ ergs,

$$dX/dt = -1{\cdot}9L_n/(6{\cdot}6 \times 10^{18}\mathcal{M}) = -2{\cdot}85 \times 10^{-19}L_n/\mathcal{M}, \tag{11}$$

where the factor $1{\cdot}9$ is to convert $L_n/\mathcal{M}$ to solar units.

Eliminating L from (3) and (7) and using (6) gives

$$R^{16\cdot5} = 6{\cdot}9 \times 10^6 Z^2\mathcal{M}^{10\cdot5}(1+Q)(1+X)X\mu^{6\cdot5},$$

and hence

$$\begin{aligned}16{\cdot}5\, R^{15\cdot5}(dR/dt) = (6{\cdot}9 \times 10^6 Z^2\mathcal{M}^{10\cdot5})\{&(1+Q)(1+X)6{\cdot}5X\mu^{5\cdot5}(d\mu/dt)\\ &+(1+Q)\mu^{6\cdot5}(1+2X)(dX/dt)\\ &+(1+X)X\mu^{6\cdot5}(dQ/dt)\}.\end{aligned} \tag{12}$$

Since $\mu = 1/(2X + \frac{3}{4}Y + \frac{1}{2}Z)$

and $Y = (1-Z) - X$

it follows that

$$d\mu/dt = -(\tfrac{5}{4})\mu^2\, dX/dt. \tag{13}$$

It will be seen later that X is always small (less than $0{\cdot}1$) for the cases considered. Thus because $d\mu/dt$ is of the same order as dX/dt the term involving $X\, d\mu/dt$ on the right-hand side of (12) is small compared to the term following it and may be neglected.

Using equations (3), (7), (10) and (11), putting $(1+X)$ equal to unity and changing the variable R to r where

$$r = R^{17\cdot5}, \tag{14}$$

equation (12) is reduced to

$$Cr^{37/35}Q(dQ/dr) + (1-Kr)Q + 1 = 0, \tag{15}$$

where

$$C = 8{\cdot}05 \times 10^4 X/\mathcal{M} \tag{16}$$

* This is for a polytrope of index $n = 3$. It is sufficiently accurate for the present purpose.

and
$$K = 1{\cdot}1 \times 10^{-2}/(Z^2 \mathcal{M}^{11\cdot5} \mu^{6\cdot5}) \tag{17}$$
are both functions of the hydrogen abundance.

The hydrogen abundance is given by the equation obtained by eliminating L from (3), (6) and (7),

$$\mu^{6\cdot5} X(1+X) = 1{\cdot}45 \times 10^{-7} r^{33/35} / Z^2 \mathcal{M}^{10\cdot5}(1+Q), \tag{18}$$

together with equation (9).

Putting $(1+X)$ equal to unity again, and substituting for X in (16) from (18) and putting

$$x = Kr \tag{19}$$

equation (15) becomes

$$x^2 \{Q/(1+Q)\} \, dQ/dx + (1-x)Q + 1 = 0. \tag{20}$$

The boundary conditions are $Q \to 0$, $dQ/dx \to 0$, as $x \to \infty$. For large x, $Q = x^{-1} + x^{-3} + \ldots$ gives the starting values for the integration.

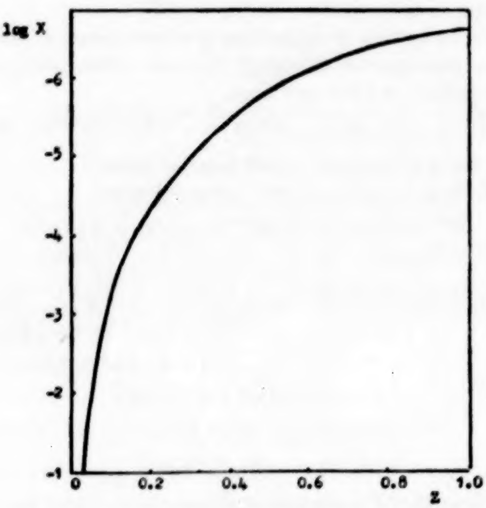

FIG. 2.—*Logarithm of hydrogen abundance against abundance of heavy elements, for the blue stars at (i) on Fig. 1.*

Assuming $X = 0$, μ is obtained from (9) and $\mathcal{M}$ from (3) for a particular value of Z. Then K is determined by (17), and x by (19) and (14) with $R = 1{\cdot}33$. Thus Q is obtained from the result of the integration of (20). X is then calculated from (8). For small Z (when $X > 10^{-2}$), μ, $\mathcal{M}$, K and Q were then recalculated. No calculations were made for values of Z less than $0{\cdot}02$ (i.e. $X > 0{\cdot}1$) since, as previously noted, the value of Z is expected to be about $0{\cdot}1$ or more.

Log X and $\mathcal{M}$ are plotted against Z in Figs. 2 and 3 respectively, and Q against Z in Fig. 4.

If the point (*i*) on Fig. 1 marks the stage of transition from nuclear to gravitational energy production (as suggested earlier in the paper), we may reasonably put $L_g = L_n$ for the stars at that point; i.e. $Q = 1$. From Figs. 2, 3 and 4 we therefore obtain for the blue stars the values

$$\mathcal{M} = 0{\cdot}93, \quad X = 5 \times 10^{-6}, \quad Z = 0{\cdot}40.$$

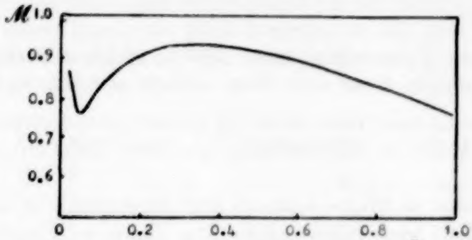

FIG. 3.—*Mass against abundance of heavy elements, for the blue stars at* (*i*) *on Fig.* 1.

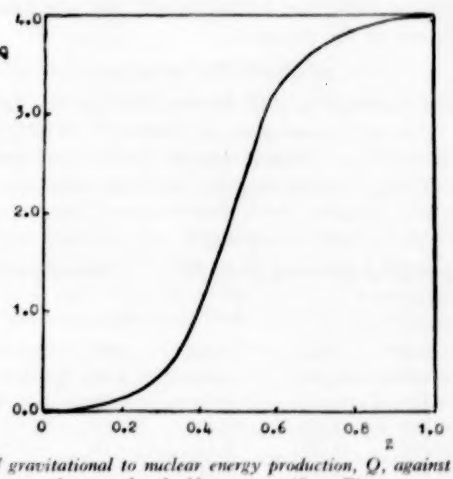

FIG. 4.—*The ratio of gravitational to nuclear energy production, Q, against abundance of heavy elements, for the blue stars at* (*i*) *on Fig.* 1.

It is seen from Fig. 3 that a considerable change in Z does not have much effect on the determined value of $\mathcal{M}$ provided Z is not below about 2 per cent. Neither does a considerable change in the value of Q employed have much effect on Z and hence on $\mathcal{M}$. The value of $\mathcal{M}$ may, however, be considerably altered by a change in R, i.e. by an error in $\log T_e$ which may result from an error in the colours of the stars or the conversion from colour to effective temperature, or by an error in the absolute magnitude.

When the action of the helium cycle in the core of a red giant has converted all the helium into carbon, oxygen and neon, the *relative* abundances by weight of the latter may be expected to be about 0·48, 0·36 and 0·16 respectively (**19**). If mixing takes place and the star becomes homogeneous, the carbon-nitrogen

cycle will be re-established in the core of the star (which will then be a blue star) and much of the carbon will be converted into nitrogen. The relative abundances of carbon and nitrogen when the carbon-nitrogen cycle proceeds under equilibrium conditions may be calculated for various temperatures (**21**), and for the temperature appropriate to this case (3×10^7 deg. K) it is found that $X_c/X_n \simeq \frac{1}{7}$. Because equilibrium conditions may not be attained this must be regarded as a limiting value. Thus the final composition of *the stars at* (*i*) on Fig. 1 may be about

60 per cent helium, 3 per cent or more carbon, 17 per cent or less nitrogen, 14 per cent oxygen, 6 per cent neon, calcium and heavier elements.

It must be emphasized that these values are subject to considerable uncertainty, because little is yet known of the conditions prevailing in the interiors of red giant stars.

RR Lyrae variables in M92.—Colours and magnitudes of six short-period variables in M92 have been published by Arp, Baum and Sandage (**22**). The colours enable the effective surface temperatures and bolometric corrections to be obtained, and if the mean absolute visual magnitude of the variables is assumed to be $+0.1 \pm 0.2$ (**6**), the absolute bolometric magnitudes may then be determined. Hence the radii may be calculated, and the masses obtained from pulsation theory by use of the equation

$$P\sqrt{(\mathcal{M}/R^3)} = 0.041, \qquad (21)$$

where the numerical constant is that derived by Epstein (**23**), and $\mathcal{M}$ and R are in solar units. The results are given in Table I. (The probable error of an individual $\log \mathcal{M}$ is ± 0.195. This is derived from the probable errors of the observed colours and magnitudes, and the probable error of the mean absolute

TABLE I

Logarithms of the masses of the RR Lyrae variables in M92

Star (Hach. No.)	Period (days)	$\bar{V}$	$\bar{P}-\bar{V}$	M_{bol}	$\log T_e$	$\log \mathcal{M}$
1	0.703	15.09	+0.22	+0.11	3.84	$\bar{1}.75$
2	0.644	15.08	+0.19	+0.16	3.85	$\bar{1}.83$
3	0.637	15.29	+0.12	+0.20	3.88	$\bar{1}.55$
5	0.620	15.07	+0.16	+0.03	3.86	$\bar{1}.80$
8	0.402	15.16	+0.23	+0.18	3.83	0.27
10	0.377	15.05	+0.15	−0.01	3.86	0.25

magnitude. It assumes no errors in the periods, in the bolometric corrections, in the conversions from colour to effective temperature, or in the value of the constant 0.041.) The mean of the values gives $\overline{\log \mathcal{M}} = -0.09 \pm 0.08$ (p.e.) and hence

$$\mathcal{M} = 0.98 \pm 0.08 \text{ (p.e.)}. \qquad (22)$$

Longer period variables in globular clusters.—Colours and magnitudes of some of the longer-period Population II variables may be obtained from colour-magnitude diagrams obtained by Arp and published by W. Baade (**24**). They are listed in Table II; the periods are those given by Helen B. Sawyer (**25**) (the stars were identified by making use of the differences in magnitude between the RR Lyrae variables and the longer-period variables, given by Sawyer).

The masses, determined in the same manner as for the RR Lyrae variables, are also listed in Table II. The mean of the values gives $\overline{\log \mathscr{M}} = -0.37 \pm 0.07$. Because of the uncertainties in the effective temperatures and bolometric corrections for large colour indices, the mean mass

$$\mathscr{M} = 0.50 \pm 0.04 \qquad (23)$$

cannot be considered as reliable as the probable error would indicate. It does however indicate that the masses of the longer period variables (which are mostly scattered on or above the upper giant branches on the globular cluster colour-magnitude diagrams) are, like the RR Lyrae variables and the blue stars, rather less than the solar mass.

Table II

Logarithms of the masses of some of the longer-period variables in globular clusters

Cluster	Period (days)	M_v	*C.I.*	M_{bol}	$\log T_e$	$\log \mathscr{M}$
M2	17·5	−2·8	+0·40	−3·0	3·75	$\bar{1}$·56
	33·6	−3·8	+0·26	−3·9	3·81	$\bar{1}$·18
M3	103·2	−3·2	+1·5	−4·0	3·55	$\bar{1}$·60
	15·3	−3·3	+0·33	−3·4	3·78	$\bar{1}$·74
M5	106·0	−3·1	+1·68	−4·1	3·5	0·2
M13	5·1	−1·8	+0·21	−1·8	3·85	$\bar{1}$·32
M15	1·44	−0·9	+0·12	−1·1	3·88	$\bar{1}$·81

Sub-dwarfs.—The exact nature of the main sequences of globular clusters is still open to doubt, but it is evident from Fig. 1 that the sub-dwarf sequence lies close to them, and may be the same.

It is possible that the absence of a Hertzsprung Gap on the globular cluster colour-magnitude diagrams indicates the absence of convective cores in the Population II main-sequence stars, which in turn suggests that energy is produced by the proton-proton reaction (**7**). If the abundance of the Russell mixture in these stars is only about a tenth of that in Population I stars, while the hydrogen abundance is high (**14, 15**), electron scattering will be of primary importance in the opacity. It is also very probable that deep hydrogen convection zones will exist in their outer layers (**26**). Although no stellar models have yet been constructed which include all these requirements, it is possible to obtain an indication of the masses of the brightest sub-dwarfs by using stellar models which include some, though not all, of the features described above (**26, 27, 20**), and taking into account the course of evolution likely to be followed by them (**28, 9**). For instance, using Henrich's stellar models with opacity due to electron scattering (but with convective cores) (**27**), and energy generation by the proton-proton reaction (**10**), and allowing for the increase in brightness by about a magnitude as the hydrogen in the inner parts of the stars is converted into helium (**28**), a value of about 0·7 solar masses is obtained for the brightest stars on the Population II main sequence (**29**). The replacement of a central convective core by an outer convective region, and the inclusion of photoelectric absorption in the opacity, will of course affect this result, though probably only by something of the order of a tenth of a solar mass.

Conclusions.—The above results, together with those of other investigators, are summarized on Fig. 1. All the individual results must be considered rather uncertain. Determinations based on theories of stellar structure and evolution are, of course, subject to uncertainties in the theories as well as in the observational

data, while the difference between the two separate determinations of the mass of Algol 2 indicates the uncertainty in calculations based on the observations and dynamics of multiple star systems. However, when all the results are considered together, they indicate that the masses of the stars of Population II increase from about 0·7 solar masses in the case of the brightest sub-dwarfs to about 1·0 solar masses for the red giants, variables, blue stars and nuclei of planetary nebulae.

The writer is indebted to Dr P. A. Sweet for many valuable discussions on various aspects of stellar structure and stellar evolution.

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ON THE DAMPING OF THE FREE NUTATION OF THE EARTH

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Summary

The free (420^d) nutation is known to be heavily damped, the characteristic damping time being less than ten periods. The origin of this damping is unknown, but must be due to dissipative, non-rigid-body movements of the Earth. The view, which has been widely held, that this may arise from the relative motion between the liquid core of the Earth and the mantle is shown to be based on an error. When account is taken of its small moment of inertia, the core cannot be held responsible for this damping, nor can a limit be derived for its viscosity. This confines the origin of the damping to a non-elastic behaviour of the mantle.

The irregular motion of the Earth relative to the direction of its axis of rotation is determined by observations of the " variation of latitude ". One component of this motion has a period of 420 days approximately, which has been recognized as the " free " or " Eulerian " nutation. For a rigid body of moments of inertia A, A, C, spinning round an axis near that of C, the period is given by $\tau A/(C-A)$ where τ is the period of rotation. $A(C-A)$ for the Earth is known accurately from the precession as 305 ; but the nutation period of 420 days can be adequately accounted for by the lengthening resulting from elastic deformation of the Earth (1). The internal stresses associated with the nutation are quite different from those associated with the precession, so that any deformation under stress will affect the two motions differently. The free nutation provides the best observational material that we have relating to the behaviour of the Earth under a large-scale and slowly changing stress system, and it may hence be of great interest in connection with the investigation of slow changes in the figure of the Earth.

Analysis of the observations of the variation of latitude is difficult. A regular annual term, thought to be a forced motion largely due to the movement of air masses (1), can be taken out, but the remaining motion is irregular in amplitude and phase. A period of approximately 420 days can be recognized ; and, on the assumption that the motion is due to random exciting impulses, it is possible to say that the damping is less than critical but more rapid than would make the amplitude of the undisturbed motion decay to $1/e$ in 10 periods.*

Geophysical problems are associated both with the explanation of the excitation of the motion and with its damping. It seems legitimate to split up the problem into these two parts, and thereby to ignore the possibility that the

* Jeffreys has quoted about 13 periods (1). More recently Walker and Young (2) have given an estimate of three periods.

damping also is due to exciting impulses, only of a non-random sort. This usual approach will be adopted in the present paper, which is restricted to the problem of the damping.

The core of the Earth, assumed liquid from the seismic information regarding the absence of shear waves, has sometimes been considered as the probable, or at any rate possible, source of the damping (1). Also a limiting value of the viscosity of the liquid has been derived, and a greater viscosity than this has been considered ruled out as it would cause excessive damping (1).

Whilst the core can cause some damping of the motion, through the occurrence of differential motion and viscous dissipation, it will be shown that its moment of inertia is in fact not adequate to cause the observed amount, whatever the assumed value of the viscosity may be.

For this purpose we consider the following problem: What is the maximum damping that can be achieved with any method of coupling whatever between core and mantle, given only the limitation of the moment of inertia of the entire core? The only reaction to the coupling forces is provided by the inertia of the core, and it therefore appears that the greatest dissipation of energy is made possible when the entire available inertia of the core is brought to bear. This would be the case for a core constrained to move like a rigid body. We consider therefore that the problem as stated can be treated by confining attention to the case of a rigid core moving in the mantle, with arbitrary dissipative forces acting between them. The inclusion of non-dissipative passive forces (constraints) would only diminish relative motions and therefore diminish the energy that can be dissipated. The elastic behaviour of the mantle has only an indirect effect on the dissipation in this case, and so the mantle will in the first instance be considered as rigid, and the effects of its elasticity will be considered later.

A spherical core will be assumed, but this enters into the dynamical considerations only through the equality of the moments of inertia, and not in any critical way. The nature of the dissipative coupling does not require to be defined.

This treatment avoids the hydrodynamic problem of the motions in the core which is not soluble in its entirety; but it can give only an upper limit for the dissipation which must be somewhat too high, by an amount that cannot be derived by any considerations of similar simplicity.

It might be thought that a hydrodynamic treatment simplified by the use of boundary layer theory provides another possible line of attack. However, it suffers from the disadvantage that it breaks down for the case of greatest damping (and hence greatest interest) which occurs when the thickness of the boundary layer becomes comparable with the radius of the core.

Consider a shell of moments of inertia A, A, C containing in a spherical cavity a rigid sphere of moments of inertia D, D, D, the centres of mass being coincident. Axes in the shell will be used, axis 3 being along the principal axis of inertia of moment C. Let $\boldsymbol{\omega}=(\omega_1, \omega_2, \omega_3)$ be the angular velocity of the shell and $\boldsymbol{\Omega}=(\Omega_1, \Omega_2, \Omega_3)$ the angular velocity of the sphere. If the frictional couple is λ times the differential angular velocity, then the equations of motion are

$$\left.\begin{aligned} A\dot{\omega}_1+(C-A)\omega_2\omega_3 &= \lambda(\Omega_1-\omega_1) = -D[\dot{\Omega}_1+\omega_2\Omega_3-\omega_3\Omega_2], \\ A\dot{\omega}_2-(C-A)\omega_3\omega_1 &= \lambda(\Omega_2-\omega_2) = -D[\dot{\Omega}_2+\omega_3\Omega_1-\omega_1\Omega_3], \\ C\dot{\omega}_3 \qquad\qquad\quad &= \lambda(\Omega_3-\omega_3) = -D[\dot{\Omega}_3+\omega_1\Omega_2-\omega_2\Omega_1]. \end{aligned}\right\} \qquad (1)$$

If the motion differs only slightly from rigid-body rotation about the third axis of coordinates with angular velocity n, then ω_1, ω_2, $\tilde{\omega}_3$, Ω_1, Ω_2, $\tilde{\Omega}_3$ are all small, where $\tilde{\omega}_3 = \omega_3 - n$, $\tilde{\Omega}_3 = \Omega_3 - n$.

Neglecting products of small quantities, equations (1) become

$$\left.\begin{aligned} A\dot{\omega}_1 + (C-A)n\omega_2 &= \lambda(\Omega_1 - \omega_1) = -D[\dot{\Omega}_1 + n(\omega_2 - \Omega_2)], \\ A\dot{\omega}_2 - (C-A)n\omega_1 &= \lambda(\Omega_2 - \omega_2) = -D[\dot{\Omega}_2 - n(\omega_1 - \Omega_1)], \\ C\dot{\tilde{\omega}}_3 \qquad &= \lambda(\tilde{\Omega}_3 - \tilde{\omega}_3) = -D\dot{\tilde{\Omega}}_3. \end{aligned}\right\} \tag{2}$$

The third equation is independent and does not involve any quantities of interest for the present investigation, and will therefore not be considered any further.

Let

$$\left.\begin{aligned} \omega_1 + \mathrm{i}\,\omega_2 &= \omega, & \lambda &= nD\psi, \\ \Omega_1 + \mathrm{i}\,\Omega_2 &= \Omega, & C &= A(1+\beta), \\ & & D &= A\alpha. \end{aligned}\right\} \tag{3}$$

Then the first pair of equations (2) may be written

$$\frac{1}{n}\dot{\omega} - \mathrm{i}\beta\omega = \alpha\psi(\Omega - \omega) = -\alpha\left[\frac{1}{n}\dot{\Omega} - \mathrm{i}(\omega - \Omega)\right]. \tag{4}$$

To examine the normal modes of the system, suppose that ω and Ω are proportional to $\exp(\mathrm{i}qnt)$ and that $\Omega = \kappa\omega$, where q and κ are complex numbers to be determined. Then

$$\mathrm{i}(q-\beta) = \alpha\psi(\kappa - 1) = -\mathrm{i}\,\alpha[\kappa q - 1 + \kappa]. \tag{5}$$

Accordingly

$$\kappa = 1 + \frac{\mathrm{i}}{\alpha\psi}(q - \beta) \tag{6}$$

and

$$q^2 + q[1 - \beta - \mathrm{i}\psi(1+\alpha)] - \beta(1 - \mathrm{i}\psi) = 0. \tag{7}$$

In our problem, β has a value of approximately 0·003, while α is about 0·1, and ψ is an almost unknown quantity. It is most convenient to proceed by obtaining the roots for small β of the quadratic equation (7). These are

$$q = \beta\frac{1 - \mathrm{i}\psi}{1 - \mathrm{i}\psi(1+\alpha)} \quad \text{with} \quad \kappa = 1 - \frac{\beta}{1 - \mathrm{i}\psi(1+\alpha)} \tag{8}$$

and

$$q = -1 + \mathrm{i}\psi(1+\alpha) - \beta\frac{\mathrm{i}\psi\alpha}{1 - \mathrm{i}\psi(1+\alpha)} \quad \text{with} \quad \kappa = -\frac{1}{\alpha} - \frac{\mathrm{i}}{\alpha\psi} - \frac{\mathrm{i}\beta}{\alpha\psi}\frac{1 - \mathrm{i}\psi}{1 - \mathrm{i}\psi(1+\alpha)}. \tag{9}$$

The mode corresponding to root (8) is the Eulerian nutation. The variation of q and κ in their complex planes for variable ψ is shown in Figs. 1 and 2. For zero ψ the core is of course completely independent of the shell and so does not partake of the nutation. It hence continues to spin about the fixed direction given by the angular momentum of the shell. As the angle between the angular momentum and the angular velocity of the shell is β times the angle between the angular velocity and the axis of symmetry, the core has, in our coordinates, a transverse angular velocity equal to $(1-\beta)$ times that of the shell. For large ψ the angular velocities of shell and core approach each other and the frequency of the nutation changes from βn (the value for the Eulerian nutation of the shell alone) to $\beta n/(1+\alpha)$, the value for the combined rigid body. The damping has a maximum of

$$n\beta\frac{\alpha}{2(1+\alpha)} \quad \text{for} \quad \psi = \frac{1}{1+\alpha}. \tag{10}$$

The mode described by (9) is entirely different. As $\psi \to 0$, $\kappa \to \infty$ and $q \to -1$. For vanishing friction it therefore simply represents the rotation of the core about an axis different from the axis of the shell and fixed in space. The shell is wholly unaffected.

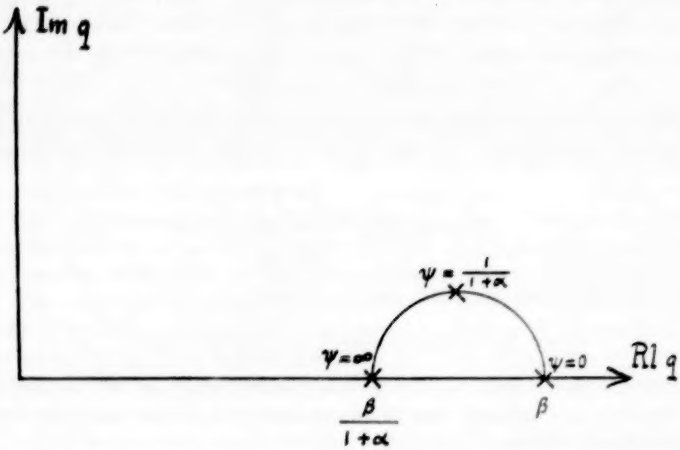

FIG. 1.—*Schematic diagram showing the variation of q in its complex plane with changing ψ.*

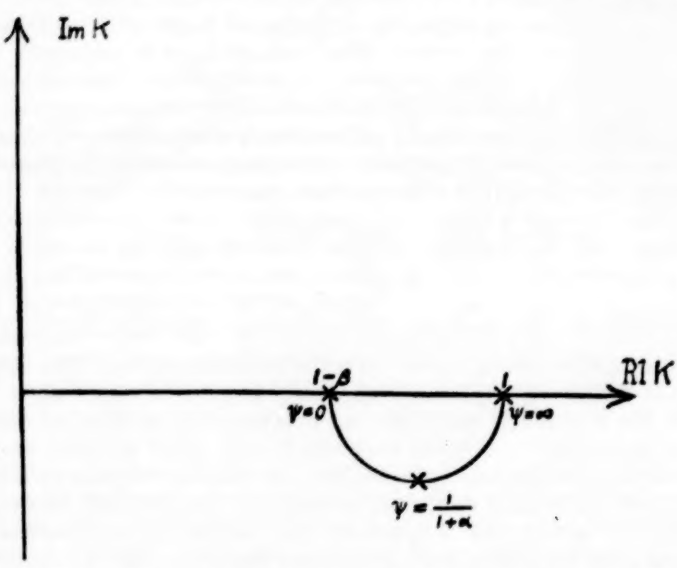

FIG. 2.—*Schematic diagram showing the variation of κ in its complex plane with changing ψ.*

As ψ increases, the real part of q changes slightly from -1 to $-1-\alpha\beta/(1+\alpha)$, while the damping increases without limit.

The main effect of this mode is that the axis of rotation of the shell gradually changes its position in space, from its original direction to that of the common

angular momentum of shell and core. This mode is certainly excited by the precession, and there may possibly be other excitations, but no effects due to this have been recognized in the observations.

Equation (10) gives the maximum damping possible with the model consisting of a rigid shell and containing a rigid but slipping core.* But before this result can be applied to the case of the Earth, to ascertain whether or not a model without a dissipative mantle can account for the observed damping, corrections ought to be made for two known inadequacies of the model, namely the elastic deformation of the mantle and the fluidity of the core.

The elastic deformation of the mantle is known approximately from the lengthening of the nutation period which it causes (1). A result of this deformation has been thought to be a diminution of the amplitude of nutation with depth (the motion being forced on the body of the Earth by the equatorial bulge). Jeffreys supposes this amplitude to have diminished to $\frac{1}{7}$ of the rigid-body value at the interface between core and mantle (3). If such a large reduction is appropriate, then it cannot be left out of consideration here. To obtain a rough estimate of the diminution of the maximum damping that would result using the factor $\frac{1}{7}$, consider two models similar in other respects but one possessing a rigid and the other a suitably elastic mantle. For the conditions at the core-mantle interface to be the same in the two, the amplitude of the nutation at the surface of the elastic model would have to be seven times as great as at the surface of the rigid one. The initial rate of dissipation of energy in the core must then be the same; but owing to the greater amplitude further out, the elastic model will possess a greater amount of energy. The ratio of the energies of the two models can be estimated as about 20 (it would be 49 if all the deformation occurred just outside the core-mantle interface) and therefore the initial rate of damping for the case discussed will be 20 times lower for the elastic than for the rigid model. (These statements become inaccurate if the damping time is comparable with the period.)

The fluidity of the core introduces a further correction, also in the sense of diminishing the maximum damping that is possible. For such values of the viscosity as are required for high damping, the boundary layer in the liquid becomes of a thickness comparable with the size of the core. Therefore the moment of inertia of the core effective for this motion is less than that of a rigid core of similar mass, which was used for deriving equation (10). Diminishing α there reduces the maximum damping. Again a detailed calculation is not warranted, but a correction factor of about 2 would appear to be appropriate.

The shortest possible damping time due to viscous dissipation in the core, assuming the optimum value of viscosity, would therefore be about 40 times longer than given by equation (10). The amplitude cannot then be diminished in the ratio $1/e$ in fewer than 120 periods. Since the observations indicate a very much more rapid damping ($3-13$ periods) it follows that dissipation in the core cannot be responsible; and moreover that no limit can be placed on the viscosity of the core from such considerations. The introduction of magnetic instead of viscous forces does not alter these conclusions.

Even if the reduction of amplitude in the mantle were by a factor not as large as 7, as given by Jeffreys, it would still be unlikely that agreement with observation

* It can be shown that if the core is subdivided into a smaller rigid sphere surrounded by a rigid shell slipping on both its inner and outer surface the maximum damping is reached for an indefinitely tight coupling at the inner surface. This model then reduces to the one considered here.

could be obtained. The model capable of the highest damping, namely a rigid mantle with a slipping, rigid core, would only be just inside the observational values for a narrow range of values of the frictional coupling; therefore even if one ignored the other factors one would still have to appeal to the coincidence that the actual value of the friction should just be the optimum, in order to account for the damping by core-mantle friction.*

It is therefore necessary to suppose that the dissipative action occurs in the mantle of the Earth; the atmosphere and the oceans can be excluded by similar considerations also on the grounds of their inadequate moments of inertia. This implies a behaviour of the material of the mantle which is not common amongst solids in the more familiar conditions of low temperature and zero pressure, namely the combination of high Young's modulus with very large departure from perfect elasticity for slowly varying strains.

This work is in disagreement with Jeffreys's result (1) that a certain value of the viscosity in the core would achieve the observed damping. Jeffreys's result is however erroneous. The angle between the instantaneous axis of rotation and the angular momentum vector is only $\frac{1}{420}$ of the angle between the axis of symmetry and the axis of rotation. The differential velocities given by Jeffreys should be reduced by this factor. Therefore his figure for the dissipation is too large by a factor of $(420)^2 = 1{\cdot}8 \times 10^5$.† To achieve the required damping, the viscosity must hence exceed the value calculated by Jeffreys (10^5 cm²/sec) by a factor of $(1{\cdot}8 \times 10^5)^2 = 3 \times 10^{10}$. For such a high viscosity the boundary layer would be $0{\cdot}6$ km $\times 1{\cdot}7 \times 10^5 = 10^5$ km thick, i.e. far larger than the core, so that the theory would break down.

The above correction would remove the discord between Jeffreys's boundary layer treatment and the method of this paper, but only for such values of the viscosity as make the boundary layer theory applicable (i.e. values well below those that lead to maximum damping).

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1955 *January* 3.

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References

(1) H. Jeffreys, *The Earth*, 3rd ed., Chapters 7 and 8, Cambridge, 1950.
(2) A. M. Walker and A. Young, "The Analysis of the Observations of the Variations of Latitude", in preparation.
(3) H. Jeffreys, *M.N.*, **109**, 670, 1949.

* Sir Harold Jeffreys has kindly informed us that he now considers his work leading to a substantial reduction of the amplitude of the motion at the core boundary as erroneous, and that therefore the factor should be close to unity instead of the $\frac{1}{7}$ that he has previously given (1).

† If the factor $\frac{1}{7}$ is deleted (see previous footnote) then the rate of dissipation is increased by 49 and therefore the discrepancy is reduced to a factor of $3{\cdot}8 \times 10^3$.

ON THE RELATION BETWEEN THE STREAM AND THE ELLIPSOID CONSTANTS

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Summary

The mean speeds, both in the positive and the negative directions, along and perpendicular to the axis of star-streaming are derived, both on the two-drifts and the ellipsoidal theories of the distribution of stellar linear velocities, the origin of velocity components being the centre of rest of all stars.

It is then found that, if the numbers of stars belonging to each drift are equal, it is possible to calculate the ratio of the axes of the velocity ellipsoid from the relative speeds of the two drifts.

The relation thus obtained is applied to the results of a number of analyses of proper motions and the calculated values of the axis-ratio are compared with the corresponding values derived from the analyses of the same material on the ellipsoidal hypothesis.

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1. An important feature of the results of analyses of the proper motions of the stars is the agreement in the direction of the vertex of star-streaming as found by the two methods of analysis—the Two-Drifts and the Ellipsoidal. As the two methods of analysis are very different—the two-drifts analyses generally being performed by fitting theoretical curves to the observed distribution in individual areas of the sky and the ellipsoidal analyses by a purely numerical method—the importance of the agreement is enhanced. It is therefore to be expected that there should exist some relationship, possibly only approximate, between certain of the parameters of the two distributions.

One such relation, which was derived by Smart (**1**), has been used with success in a number of practical applications (**2, 3, 4**).

In theory, the use of Smart's relation is valid only when applied to restricted proper motions (those greater than a given value of μ) and when the following conditions are satisfied:—

(*a*) the space distribution of the stars is given by a density law of the form

$$N(r)=\frac{A}{r}\exp(-h^2k^2r^2) \quad (A, h, k \text{ constants}),$$

(*b*) the ratio of k to μ is small, and

(*c*) the stars are equally divided between the two drifts.

It must also be noted that the relation only applies to a small region of the sky. To obtain the constants of the velocity ellipsoid for the whole sky, the results for each region must first be combined. The manner in which this may be accomplished has been described by Smart (**2**) and is basically one of successive approximations to the true values of the unknowns from initially assumed values

of the unknowns. In this paper an attempt is made to derive a more general relation.

2. In the two-drifts theory, the stars are divided into two assemblies, for each of which the distribution of linear velocities is random. The two drifts are intermingled in space and their centres of rest are in relative motion along the axis of star-streaming, this latter axis defining the direction of the vertex.

From the centre of rest of all stars the axis of star-streaming appears as an axis of preferential motion, i.e. one of greater mobility than in any other direction. In Schwarzschild's ellipsoidal theory this feature is expressed analytically. The exponent of the distribution is of the form of the quadratic expression appearing in the equation of an ellipsoid of revolution and the mean velocity component in the direction of the vertex is greater than in any other direction.

Throughout this paper the following forms of the distribution functions will be used. For a single drift

$$f(u, v, w) = \frac{nh^3}{\pi^{3/2}} \exp[-h^2(u^2+v^2+w^2)],$$

the origin of velocity components u, v and w being the centre of rest of the drift, n the number of stars belonging to the drift and $h(=1/s\sqrt{2})$ a constant, s being the velocity dispersion of stars in the drift. Thus for two drifts we have

$$f'(u, v, w) = \frac{n_1h_1^3}{\pi^{3/2}} \exp[-h_1^2(u_1^2+v_1^2+w_1^2)] + \frac{n_2h_2^3}{\pi^{3/2}} \exp[-h_2^2(u_2^2+v_2^2+w_2^2)],$$

where u_1, v_1, w_1, n_1, h_1 and u_2, v_2, w_2, n_2 and h_2 refer to Drifts I and II respectively. As the distribution of velocities in each drift is random, we may orient the axes in any convenient manner. We will take the u_1 and u_2 axes to be along the axis of star-streaming, found, in practice, to be in, or near, the galactic plane. The other axes are perpendicular to these, the w_1 and w_2 axes being approximately in the direction of the galactic pole.

We now take new axes (U, V, W) referred to the centre of rest of all stars and parallel to the u_1, v_1 and w_1 axes. Then, if the speeds of Drift I and Drift II, relative to the centre of rest of all stars, are Q_1 and Q_2 respectively, we have

$$n_1Q_1 = n_2Q_2 \tag{2.1}$$

and

$$\left.\begin{aligned} u_1 &= U - Q_1, \quad v_1 = V, \quad w_1 = W, \\ u_2 &= U + Q_2, \quad v_2 = V, \quad w_2 = W. \end{aligned}\right\} \tag{2.2}$$

The distribution function for the two-drifts theory then becomes

$$\begin{aligned} \pi^{3/2}F_1(U, V, W) = {} & n_1h_1^3 \exp[-h_1^2(U-Q_1)^2 - h_1^2(V^2+W^2)] \\ & + n_2h_2^3 \exp[-h_2^2(U+Q_2)^2 - h_2^2(V^2+W^2)]. \end{aligned} \tag{2.3}$$

For the ellipsoidal theory the distribution function, referred to the same axes, is

$$\pi^{3/2}F_2(U, V, W) = nKH^2 \exp[-K^2U^2 - H^2(V^2+W^2)], \tag{2.4}$$

where $n = n_1 + n_2$ and $K < H$.

3. Let the number of stars on the two-drifts theory, with $U > 0$, be

$$N' = N_1' + N_2',$$

where N_1' is the number of such stars belonging to Drift I and N_2' is the corresponding number for Drift II, for all values of V and W.

$$\therefore\ N_1' = \frac{n_1 h_1^3}{\pi^{3/2}} \int_0^\infty \exp[-h_1^2(U-Q_1)^2]\,dU \iint_{-\infty}^{\infty} \exp[-h_1^2(V^2+W^2)]\,dV\,dW$$

$$= \frac{n_1 h_1}{\sqrt{\pi}} \int_0^\infty \exp[-h_1^2(U-Q_1)^2]\,dU. \tag{3.1}$$

Let $x = h_1(U-Q_1)$ and $\tau_1 = h_1 Q_1$.

Then we have
$$N_1' = \frac{n_1}{\sqrt{\pi}} \int_{-\tau_1}^\infty \exp(-x^2)\,dx.$$

Define
$$\Theta(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^\alpha \exp(-x^2)\,dx. \tag{3.2}$$

Then
$$N_1' = \frac{n_1}{2}\{1+\Theta(\tau_1)\}. \tag{3.3}$$

Similarly
$$N_2' = \frac{n_2}{2}\{1-\Theta(\tau_2)\}, \tag{3.4}$$

where $\tau_2 = h_2 Q_2$.

$$\therefore\ N' = \frac{n}{2} + \tfrac{1}{2}\{n_1\Theta(\tau_1) - n_2\Theta(\tau_2)\}. \tag{3.5}$$

By the same methods, if $N'' = N_1'' + N_2''$ be the number of stars with $U < 0$ for all values of V and W, then

$$N'' = \frac{n}{2} - \tfrac{1}{2}\{n_1\Theta(\tau_1) - n_2\Theta(\tau_2)\}. \tag{3.6}$$

Put
$$G(\tau_1,\tau_2) = n_1\Theta(\tau_1) - n_2\Theta(\tau_2). \tag{3.7}$$

Then
$$N' = \frac{n}{2} + \tfrac{1}{2}G(\tau_1,\tau_2),$$

$$N'' = \frac{n}{2} - \tfrac{1}{2}G(\tau_1,\tau_2) = N' - G(\tau_1,\tau_2). \tag{3.8}$$

On the ellipsoidal theory we have

$$N' = N'' = \frac{nK}{\sqrt{\pi}} \int_0^\infty \exp(-k^2U^2)\,dU = \frac{n}{2}. \tag{3.9}$$

4. If we write U' for the mean speed in the positive direction of the U-axis, then on the two-drifts theory we have:

$$N'U' = \frac{n_1 h_1}{\sqrt{\pi}} \int_0^\infty U \exp[-h_1^2(U-Q_1)^2]\,dU + \frac{n_2 h_2}{\sqrt{\pi}} \int_0^\infty U \exp[-h_2^2(U+Q_2)^2]\,dU. \tag{4.1}$$

Writing τ_1 for $h_1 Q_1$ and τ_2 for $h_2 Q_2$ as above, and putting $x = h_1(U-Q_1)$ and $y = h_2(U+Q_2)$, we have, in (4.1),

$$N'U' = \frac{n_1}{h_1\sqrt{\pi}} \int_{-\tau_1}^\infty (x+\tau_1)\exp(-x^2)\,dx + \frac{n_2}{h_2\sqrt{\pi}} \int_{\tau_2}^\infty (y-\tau_2)\exp(-y^2)\,dy$$

$$= \frac{n_1}{2h_1\sqrt{\pi}}[\exp(-\tau_1^2) + \tau_1\sqrt{\pi}\{1+\Theta(\tau_1)\}]$$

$$+ \frac{n_2}{2h_2\sqrt{\pi}}[\exp(-\tau_2^2) - \tau_2\sqrt{\pi}\{1-\Theta(\tau_2)\}].$$

Rewriting (2.1) we obtain
$$\frac{n_1\tau_1}{h_1} = \frac{n_2\tau_2}{h_2}.$$

$$\therefore \quad N'U' = \frac{n_1}{2h_1\sqrt{\pi}}\{\exp(-\tau_1^2)+\tau_1\sqrt{\pi}\Theta(\tau_1)\} + \frac{n_2}{2h_2\sqrt{\pi}}\{\exp(-\tau_2^2)+\tau_2\sqrt{\pi}\Theta(\tau_2)\}.$$

$$\therefore \quad U' = \frac{\dfrac{n_1}{h_1\sqrt{\pi}}\{\exp(-\tau_1^2)+\tau_1\sqrt{\pi}\Theta(\tau_1)\} + \dfrac{n_2}{h_2\sqrt{\pi}}\{\exp(-\tau_2^2)+\tau_2\sqrt{\pi}\Theta(\tau_2)\}}{n+G(\tau_1,\tau_2)}. \tag{4.2}$$

Similarly, if U'' is the mean speed in the negative direction of the U-axis, then, by the above methods,

$$U'' = \frac{\dfrac{n_1}{h_1\sqrt{\pi}}\{\exp(-\tau_1^2)+\tau_1\sqrt{\pi}\Theta(\tau_1)\} + \dfrac{n_2}{h_2\sqrt{\pi}}\{\exp(-\tau_2^2)+\tau_2\sqrt{\pi}\Theta(\tau_2)\}}{n+G(\tau_1,\tau_2)-2G(\tau_1,\tau_2)}. \tag{4.3}$$

For convenience we may write (4.2) as

$$U' = \frac{\phi\{n_1, h_1, \tau_1;\, n_2, h_2, \tau_2\}}{\psi(n, \tau_1, \tau_2)} = \frac{\phi}{\psi}; \tag{4.4}$$

(4.3) then becomes

$$U'' = \frac{\phi}{\psi - 2G(\tau_1, \tau_2)}. \tag{4.5}$$

The expressions for the mean speeds U' and U'' may also be derived on the ellipsoidal theory. In this case we obtain

$$U' = U'' = 1/K\sqrt{\pi}. \tag{4.6}$$

5. We have derived the facts (i) that on the two-drifts theory the number of stars with $U>0$ can differ from the number with $U<0$, and (ii) that the mean speed in the positive direction of the U-axis can differ from that in the negative direction. On the ellipsoidal theory these differences do not occur. Thus the first conditions for possible identity of the two theories are those which we shall obtain from equating N' to N'' and U' to U'' on the two-drifts theory. From (3.8) we have

$$N'' = N' - G(\tau_1, \tau_2),$$

whence for N'' to equal N' we must have

$$G(\tau_1, \tau_2) = 0. \tag{5.1}$$

Also, from (4.4) and (4.5), for U' to equal U'', we must again have the above condition. This is thus the primary condition for relating the two theories. Writing it out fully, it is

$$n_1\Theta(\tau_1) = n_2\Theta(\tau_2). \tag{5.2}$$

Substituting for τ_2 in (5.2) from (2.1) and putting $\alpha = n_1/n_2$ and $\beta = h_2/h_1$ we obtain

$$\alpha\Theta(\tau_1) = \Theta(\alpha\beta\tau_1). \tag{5.3}$$

Now in all practical applications of the two-drifts theory it is assumed that $h_1 = h_2 = h$ say, where $1/h$ is usually defined to be the theoretical unit of velocity. Hence in (5.3) we may write $\beta = 1$. Then on expanding the function $\Theta(\tau_1)$, (5.3) becomes

$$\alpha\int_0^{\tau_1}\exp(-x^2)\,dx = \int_0^{\alpha\tau_1}\exp(-y^2)\,dy.$$

We now put $x = \tau_1 T$ and $y = \tau_1 t$. The condition now becomes

$$\alpha\tau_1\int_0^1\exp(-\tau_1^2T^2)\,dT = \alpha\tau_1\int_0^1\exp(-\alpha^2\tau_1^2t^2)\,dt.$$

$$\therefore \quad \alpha = 1. \tag{5.4}$$

Thus the condition governing the relation of the two theories is that, as h_1 is taken to be equal to h_2, we must have $n_1 = n_2$, and hence, from (2.1), $\tau_1 = \tau_2 = \tau$ say.

We then have
$$U' = U'' = \frac{\exp(-\tau^2) + \sqrt{\pi} \, . \, \tau \, . \, \Theta(\tau)}{h\sqrt{\pi}} \tag{5.5}$$
and
$$N' = N'' = n/2. \tag{5.6}$$

Now the number of stars which, for all values of U, have velocities between V and $V+dV$ and W and $W+dW$, on the two-drifts theory is
$$\mathcal{N} = \frac{n_1 h_1^2}{\pi} \exp[-h_1^2(V^2+W^2)]\, dV\, dW + \frac{n_2 h_2^2}{\pi} \exp[-h_2^2(V^2+W^2)]\, dV\, dW.$$

But we have just shown that in order to relate this result to the corresponding result given by the ellipsoidal theory we must take $h_1 = h_2 = h$ and $n_1 = n_2 = n/2$.
$$\therefore \mathcal{N} = \frac{nh^2}{\pi} \exp[-h^2(V^2+W^2)]\, dV\, dW. \tag{5.7}$$

On the ellipsoidal theory
$$\mathcal{N} = \frac{nH^2}{\pi} \exp[-H^2(V^2+W^2)]\, dV\, dW. \tag{5.8}$$

It therefore follows that, to relate the two-drifts theory to the ellipsoidal, we must identify h with H, that is,
$$h = H. \tag{5.9}$$

(5.5) now becomes
$$U' = U'' = \frac{\exp(-\tau^2) + \tau\sqrt{\pi}\Theta(\tau)}{H\sqrt{\pi}}. \tag{5.10}$$

Hence, as from (4.6), we have, on the ellipsoidal theory,
$$\frac{K}{H} = \frac{1}{\exp(-\tau^2) + \tau\sqrt{\pi}\Theta(\tau)} = M(\tau). \tag{5.11}$$

We have thus derived an expression for the ratio of the axes of the velocity ellipsoid in terms of the relative speeds of the two drifts. It must be noted that this relation does not imply that an analysis of proper motions on the ellipsoidal theory will yield an axis-ratio equal to that given by (5.11) from a two-drifts analysis of the same material. The value given by (5.11) is the axis-ratio of the velocity ellipsoid that will yield the same values for the mean speeds along the axis of star-streaming, that is, it is the axis-ratio of the ellipsoid "equivalent" to the two-drifts solution.

As with Smart's formula, it is strictly applicable only when the stars are equally divided between the two drifts. The relation is not, however, limited to a region of the sky, as is Smart's formula, but applies to the whole sky. It is easily shown, however, by a similar analysis, that the relation also applies to a region of the sky. Here, if hV_1, θ_1, n_1; hV_2, θ_2 and n_2 are the drift constants of a region, then the local value of τ (τ' say), is given by
$$2\tau' = \{(hV_1 \sin\theta_1 - hV_2 \sin\theta_2)^2 + (hV_1 \cos\theta_1 - hV_2 \cos\theta_2)^2\}^{1/2}. \tag{5.12}$$

By using the value of τ' derived above in (5.11), the local value of the axis-ratio (usually written as k/h) can be found. The position angle of the vertex in the region can also be determined from the local drift constants as, if it is θ_v, then:
$$\tan\theta_v = \frac{hV_1 \sin\theta_1 - hV_2 \sin\theta_2}{hV_1 \cos\theta_1 - hV_2 \cos\theta_2}. \tag{5.13}$$

Thus, from (5.12) and (5.13), we may calculate the constants of the velocity ellipsoid from the stream analysis of a region, it being assumed that there are equal numbers of stars in each drift. These can then be combined directly to give the vertex direction and the axis-ratio, without using the method of successive approximations given by Smart.

It also might be noted that if the two-drifts and the ellipsoidal distributions are to have the same axes, then Schwarzschild's form of the ellipsoidal theory must be used. This follows from (5.7), (5.8) and (5.9).

6. In Table I, below, the values of the function $M(\tau)$ are given for a range of values of τ, the range covering the values obtaining in most investigations. In Table II the values of K/H, calculated from formula (5.11), for a number of two-drifts analyses, are compared with the values of K/H, derived from analyses of the

TABLE I

τ	K/H	τ	K/H	τ	K/H
0·0	1·0000	0·5	0·8064	1·0	0·5372
0·1	0·9902	0·6	0·7463	1·1	0·4964
0·2	0·9618	0·7	0·6880	1·2	0·4602
0·3	0·9178	0·8	0·6331	1·3	0·4280
0·4	0·8652	0·9	0·5827	1·4	0·3994

TABLE II

Source	Sp. group	N_1/N_2	2τ	K/H (obs.)	K/H (calc.)	Smart's formula
P.G.C.	ALL	1·5	1·868	0·52	0·567	
P.G.C.	ALL		1·924 ± 0·135	0·48	0·554 ± 0·031	
P.G.C.	ALL		1·962 ± 0·053	0·52	0·546 ± 0·011	
G.C.	F	1·22	2·075 ± 0·039	0·574	0·519 ± 0·008	
G.C.	K	1·02	1·638 ± 0·038	0·67	0·623 ± 0·010	
G.C.	F–M	1·19	1·816 ± 0·030	0·62	0·579 ± 0·007	
C1	F	1·17	1·873 ± 0·05	0·552	0·565 ± 0·012	0·62
C1	G	1·04	1·756 ± 0·05	0·56	0·591 ± 0·024	0·63
C1	K	1·08	1·606 ± 0·06	0·65	0·632 ± 0·015	0·67
C1	ALL	1·41	1·638 ± 0·04	0·625	0·623 ± 0·011	0·70
C2	ALL	1·71	1·648 ± 0·04	0·602 ± 0·03	0·621 ± 0·009	
C2	F	1·78	1·639 ± 0·04	0·676 ± 0·05	0·622 ± 0·012	
C2	G	1·47	1·697 ± 0·08	0·554 ± 0·04	0·608 ± 0·020	
C2	K	1·40	1·601 ± 0·08	0·606 ± 0·03	0·633 ± 0·020	
C2	F–M	1·50	1·618 ± 0·10	0·601 ± 0·03	0·629 ± 0·030	
C3	F	1·19	1·195 ± 0·15	0·775 ± 0·030	0·748 ± 0·045	
C3	G	0·97	1·040 ± 0·12	0·751 ± 0·015	0·788 ± 0·034	
C3	K	1·00	0·820 ± 0·11	0·898 ± 0·021	0·859 ± 0·031	
C3	F–M	0·99	0·985 ± 0·12	0·825 ± 0·024	0·811 ± 0·035	

It must be mentioned here that the groupings used in the P.G.C. and the G.C. analyses were not identical.

same material, by Schwarzschild's automatic method. The first column in the table gives the sources of the figures in each row. The second column gives the spectral group involved. In this column the letter F represents the grouping A5–F5, G the group F8–G5, K the group K0–M, F–M the group A5–M and ALL normally is the group B8–M.

In the third column is given the value of the ratio of the number of stars belonging to each drift—in the sense N_1 to N_2. In the fourth, fifth and sixth columns are given, respectively, the values, together with their probable errors, where available, of the relative speeds of the drifts, the observed values of K/H,

and the calculated values. For the two-drifts analyses, the values of 2τ are listed, this being the quantity usually derived.

The following sources have been used in compiling this table:

Source	Designation	Investigators Two-Drift	Investigators Ellipsoidal
Boss Preliminary General Catalogue	P.G.C.	Eddington (**5**) Jones (**6**)	Raymond (**7**)
Boss General Catalogue	G.C.	Tannahill (**8**)	Delhaye (**9**)
Cape Astrographic Zone—Volume I	C1	Smart and Tannahill (**3**, **4**)	Jackson (**10**)
,, —Volume II	C2	Ewart (**11**)	Ewart (**11**)
Cape Photographic Zone Catalogue (Zone −30° to −40°)	C3	Ewart and Lourens (**12**)	Ewart and Lourens (**12**)

Also listed for the C1 group are the values calculated by Smart and Tannahill from Smart's formula.

From the figures in the above table, it can be thus seen that the values of K/H derived by the two methods agree well, especially when the probable errors of the determinations are considered. A point of some interest is that quite good agreement is obtained even when the ratios of the number of stars belonging to the two drifts depart considerably from the assumed value of unity. This especially is the case for the values from the second volume of the Cape Astrographic Zone catalogues. In the paper describing the analyses of these proper motions it is, however, stated that these abnormally high values are probably spurious, being occasioned by the uncertainties in the determinations of the drift constants, especially those of Drift II. The above results would seem to confirm this—that is, the drift ratio for the stars of this catalogue is probably close to unity, as was the case for the first catalogue.

The belief that the stars are equally divided between the two drifts therefore, perhaps, receives some support from these results. Unfortunately, as it is not possible to determine the probable errors of the values of the ratios of the numbers of stars in the two drifts, this question cannot be settled. The results of some recent investigations have suggested that the ratio is higher for the early-type stars, of types B8–F5, than for the later types, for which it is nearly one. The higher values for the earlier type stars, however, may be only the effects of errors.

7. The analysis given in the preceding sections was based on the assumption that the direction of the major axis of the velocity ellipsoid was identical with the direction of relative motion of the two drifts. The results of the application of the relation derived on this basis indicate that the "equivalent" ellipsoid differs little from the observed ellipsoid.

In this section the analysis in the preceding sections is extended to consider the effects of a divergence between the two directions.

As in the previous analysis the forms of the distribution functions that will be used are those referred to axes with origin at the centre of rest of all stars. We will take these axes to lie along the three axes of the velocity ellipsoid.

Then for the velocity ellipsoid we now have

$$\pi^{3/2}F_1(U, V, W) = nKHJ\exp(-K^2U^2 - H^2V^2 - J^2W^2). \qquad (7.1)$$

This is a generalized form of the function used in the previous sections.

For the two-drifts theory the distribution function will now be

$$\pi^{3/2}F_2(U, V, W) = n_1h_1^3 \exp\{-h_1^2[(U-Q_1)^2+(V-R_1)^2+(W-S_1)^2]\}$$
$$+n_2h_2^3 \exp\{-h_2^2[(U+Q_2)^2+(V+R_2)^2+(W+S_2)^2]\}. \qquad (7.2)$$

Then if N' is the number of stars with $U>0$, for all values of V and W, on the two-drifts theory, we have

$$N' = \frac{n_1h_1^3}{\pi^{3/2}}\int_0^\infty \exp[-h_1^2(U-Q_1)^2]\,dU \int_{-\infty}^{\infty} \exp[-h_1^2(V-R_1)^2]\,dV$$
$$\times \int_{-\infty}^{\infty} \exp[-h_1^2(W-S_1)^2]\,dW + \frac{n_2h_2^3}{\pi^{3/2}}\int_0^\infty \exp[-h_2^2(U+Q_2)^2]\,dU$$
$$\times \int_{-\infty}^{\infty} \exp[-h_2^2(V+R_2)^2]\,dV \int_{-\infty}^{\infty} \exp[-h_2^2(W+S_2)^2]\,dW.$$

$$\therefore \quad N' = \frac{n_1h_1}{\sqrt{\pi}}\int_0^\infty \exp[-h_1^2(U-Q_1)^2]\,dU + \frac{n_2h_2}{\sqrt{\pi}}\int_0^\infty \exp[-h_2^2(U+Q_2)^2]\,dU.$$

$$\therefore \quad N' = \frac{n_1}{2}[1+\Theta(\tau_1)] + \frac{n_2}{2}[1-\Theta(\tau_2)], \qquad (7.3)$$

where $\tau_1 = h_1Q_1$ and $\tau_2 = h_2Q_2$.

Similarly, if N_1 is the number of stars with $U<0$, for all values of V and W, we have

$$N_1 = \frac{n_1}{2}[1-\Theta(\tau_1)] + \frac{n_2}{2}[1+\Theta(\tau_2)]. \qquad (7.4)$$

In a similar manner we can obtain, if N'' is the number of stars with $V>0$, for all U and W; $N_{''}$ the number of stars with $V<0$, for all U and W; N''' the number of stars with $W>0$, for all U and V; and $N_{'''}$ the number of stars with $W<0$, for all U and V:

$$N'' = \frac{n_1}{2}[1+\Theta(\rho_1)] + \frac{n_2}{2}[1-\Theta(\rho_2)], \qquad (7.5)$$

$$N_{''} = \frac{n_1}{2}[1-\Theta(\rho_1)] + \frac{n_2}{2}[1+\Theta(\rho_2)], \qquad (7.6)$$

$$N''' = \frac{n_1}{2}[1+\Theta(\lambda_1)] + \frac{n_2}{2}(1-\Theta(\lambda_2)], \qquad (7.7)$$

$$N_{'''} = \frac{n_1}{2}[1-\Theta(\lambda_1)] + \frac{n_2}{2}[1+\Theta(\lambda_2)], \qquad (7.8)$$

where $\rho_1 = h_1R_1$, $\rho_2 = h_2R_2$, $\lambda_1 = h_1S_1$ and $\lambda_2 = h_2S_2$.

8. We now derive the mean speeds along the positive and negative directions of each axis. If we let U' be the mean speed along the positive direction of the U axis, then as in Section 4, we obtain, by exactly similar methods of analysis,

$$U' = \frac{\phi\{n_1, h_1, \tau_1; n_2, h_2, \tau_2\}}{\psi\{n, \tau_1, \tau_2\}}, \qquad (8.1)$$

and by putting

U'' = mean speed along the negative direction of the U-axis,
V' = mean speed along the positive direction of the V-axis,
V'' = mean speed along the negative direction of the V-axis,
W' = mean speed along the positive direction of the W-axis, and
W'' = mean speed along the negative direction of the W-axis,

we obtain, additionally,

$$U'' = \frac{\phi\{n_1, h_1, \tau_1 ; n_2, h_2, \tau_2\}}{\psi(n\tau_1, \tau_2) - 2G(\tau_1, \tau_2)}, \tag{8.2}$$

$$V' = \frac{\phi\{n_1, h_1, \rho_1 ; n_2, h_2, \rho_2\}}{\psi\{n, \rho_1, \rho_2\}}, \tag{8.3}$$

$$V'' = \frac{\phi\{n_1, h_1, \rho_1 ; n_2, h_2, \rho_2\}}{\psi\{n, \rho_1, \rho_2\} - 2G\{\rho_1, \rho_2\}}, \tag{8.4}$$

$$W' = \frac{\phi\{n_1, h_1, \lambda_1 ; n_2, h_2, \lambda_2\}}{\psi\{n, \lambda_1, \lambda_2\}}, \tag{8.5}$$

$$W'' = \frac{\phi\{n_1, h_1, \lambda_1 ; n_2, h_2, \lambda_2\}}{\psi\{n, \lambda_1, \lambda_2\} - 2G\{\lambda_1, \lambda_2\}}. \tag{8.6}$$

Since we have the relations

$$\frac{n_1^2\{\tau_1^2 + \rho_1^2 + \lambda_1^2\}}{h_1^2} = \frac{n_2^2\{\tau_2^2 + \rho_2^2 + \lambda_2^2\}}{h_2^2}, \tag{8.7}$$

and, as the drifts are moving in opposite directions,

$$\tau_2 = A\tau_1 ; \quad \rho_2 = A\rho_1 ; \quad \lambda_2 = A\lambda_1, \tag{8.8}$$

so, from (8.7) and (8.8), we obtain

$$\left.\begin{aligned} \tau_2 &= \frac{n_1}{n_2} \cdot \frac{h_2}{h_1} \cdot \tau_1 = \alpha\beta\tau_1, \\ \rho_2 &= \alpha\beta\rho_1, \\ \lambda_2 &= \alpha\beta\lambda_1, \end{aligned}\right\} \tag{8.9}$$

where $\alpha = n_1/n_2$ and $\beta = h_2/h_1$. The corresponding expressions on the ellipsoidal theory are

$$U' = U'' = \frac{1}{K\sqrt{\pi}}, \tag{8.10}$$

$$V' = V'' = \frac{1}{H\sqrt{\pi}}, \tag{8.11}$$

$$W' = W'' = \frac{1}{J\sqrt{\pi}}. \tag{8.12}$$

Thus the condition for relating the two theories is once again

$$G(\tau_1, \tau_2) = 0, \tag{8.13}$$

$$G(\rho_1, \rho_2) = 0, \tag{8.14}$$

and

$$G(\lambda_1, \lambda_2) = 0. \tag{8.15}$$

As we have to take $h_1 = h_2 = h$ (say), this being assumed in the two-drifts analyses, the conditions (8.13) to (8.15) reduce to $n_1 = n_2 = n/2$ and $\tau_1 = \tau_2 = \tau$ (say), $\rho_1 = \rho_2 = \rho$ (say) and $\lambda_1 = \lambda_2 = \lambda$ (say). We thus have

$$\frac{1}{K} = \frac{\exp(-\tau^2) + \tau\sqrt{\pi}\Theta(\tau)}{h}, \tag{8.16}$$

$$\frac{1}{H} = \frac{\exp(-\rho^2) + \rho\sqrt{\pi}\Theta(\rho)}{h}, \tag{8.17}$$

$$\frac{1}{J} = \frac{\exp(-\lambda^2) + \lambda\sqrt{\pi}\Theta(\lambda)}{h}. \tag{8.18}$$

The ratios of the axes are thus, taking the polar axis as the least,

$$\frac{K}{J}=\frac{\exp(-\lambda^2)+\lambda\sqrt{\pi}\Theta(\lambda)}{\exp(-\tau^2)+\tau\sqrt{\pi}\Theta(\tau)}=\frac{M(\tau)}{M(\lambda)}, \tag{8.19}$$

$$\frac{H}{J}=\frac{\exp(-\lambda^2)+\lambda\sqrt{\pi}\Theta(\lambda)}{\exp(-\rho^2)+\rho\sqrt{\pi}\Theta(\rho)}=\frac{M(\rho)}{M(\lambda)}. \tag{8.20}$$

It can easily be seen that on putting $\rho=\lambda=0$ in the above equations, then (8.19) reduces to (5.11), and (8.20) to $H=J$. Also $\exp(-\lambda^2)+\lambda\sqrt{\pi}\Theta(\lambda)$ is greater than 1 for λ greater than, or equal to, zero. Thus, if the direction of the vertex differs from one distribution to the other, then even if $H=J$, the axis-ratio K/H is greater than if the directions were identical. If the differences in direction are—in galactic coordinates—ΔG in longitude and Δg in latitude, then

$$\frac{\lambda}{(\tau^2+\rho^2)^{1/2}}=\tan\Delta g \quad\text{and}\quad \frac{\rho}{\tau}=\tan\Delta G,$$

i.e.

$$\rho=\tau\tan\Delta G, \tag{8.21}$$

$$\lambda=\tau\sec\Delta G\tan\Delta g. \tag{8.22}$$

The differences in either direction rarely exceed 5° in either coordinate, and are usually of the same order of magnitude as the probable errors in the coordinates. For $\Delta G=\Delta g=5°$ we have $\rho=0{\cdot}09\tau$ and $\lambda=0{\cdot}09\tau$. The average value of τ for the first 15 values in Table II is 0·879, which gives $K/H=0{\cdot}592$. If the directions given by each method differed by 5° in each coordinate, then τ becomes 0·871 instead of 0·879, and K/H becomes 0·597 instead of 0·592. The effect is thus negligible. Therefore, unless there exists a marked difference of direction between the results of the methods of analysis, the effect on the axis-ratio of the "equivalent" ellipsoid to the two-drifts analysis is insignificant. This is especially the case when the probable errors are considered.

The extent to which the results depend on the assumed equality of the velocity dispersions of the two drifts is, however, unknown. Analyses of proper motions by themselves do not permit the evaluation of these dispersions.

9. *Acknowledgments.*—It is a pleasure to tender my thanks to Professor W. M. Smart for suggesting this investigation and for his advice whilst it was in progress. My thanks are also due to the Department of Scientific and Industrial Research for the award of a Maintenance Grant, during the tenure of which the investigation was performed.

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1954 *December* 21.

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MONOCHROMATIC MAGNITUDES OF MARS IN 1954

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Summary

Monochromatic magnitudes of Mars were determined from spectra taken on 40 nights in 1954. The plates were measured at seven wave-lengths from λ 4050 to λ 6360. Monochromatic coefficients of phase and values of the Russell-Bond albedo are given. The work is essentially a repeat of the 1952 programme but has more weight.

Mars was again observed during the opposition of 1954 using the same apparatus as in 1952.* The following stars were used as standards: β Car, α Vir, β Cen, α Oph, ϵ Sgr, α Aql, α Pav, α PsA (used in 1952) and ϵ CMa, α Leo, δ Vel, γ Cen and α Eri (not used in 1952). The magnitudes adopted for the comparison stars were taken from Woolley, Gascoigne and A. de Vaucouleurs.† Magnitudes for λ 5430 A were found by interpolation.

The observed magnitudes of Mars have been reduced to unit distance from both Earth and Sun, and are shown in Table I.

The next step is the reduction of the magnitude at unit distance m to the magnitude at zero phase g, the phase being α and the coefficient of phase μ, according to the linear relation

$$m_\lambda = g_\lambda + \mu_\lambda \alpha.$$

This was effected by the method of least squares and the results are shown in Table II. The results found in 1952 are shown for comparison.

From Table II we see that Mars appeared fainter at opposition in 1954 than in 1952 by two-tenths of a magnitude, and that the coefficient of phase in 1954 showed less dependence with wave-length than in 1952. These changes appear to be real physical changes since they are far outside the standard errors.

We will return to this subject later and remark that the values of g_λ shown in Table II lead to values of the Russell–Bond albedo (computed *exactly* as in *M.N.*, **113**, 521, 1953) as given in Table III.

It is remarkable that the Russell–Bond albedo has hardly changed between 1952 and 1954 although the planet was two-tenths of a magnitude fainter at zero phase in the later year. This is of course because the Russell–Bond albedo is calculated from 50° phase, and from Table II we get the following calculated values $m_\lambda(50^\circ) = g_\lambda + 50\mu_\lambda$:

λ	4050	4250	4550	4945	5430	5980	6360
m_λ (50°), 1954	1·33	1·04	0·38	−0·07	−0·69	−1·36	−1·65
m_λ (50°), 1952	1·39	1·08	0·42	+0·06	−0·55	−1·38	−1·66

The residuals $m-(g+\mu\alpha)$ are plotted against time of observation in Fig. 1, the crosses being means of the three wave-lengths $\lambda\lambda$ 5430, 5980, 6360 and the circles means of the four wave-lengths $\lambda\lambda$ 4050, 4250, 4550, 4945.

* *M.N.*, **113**, 521, 1953.

† *M.N.*, **114**, 514, 1954.

TABLE I

Magnitudes of Mars reduced to unit distance from both Earth and Sun

Date 1954		$5 \log R\Delta$	4050 A	4250 A	4550 A	4945 A	5430 A	5980 A	6360 A	Phase $\lvert A_{\oplus} - A_{\odot} \rvert$ °
April	1·6	+0·846	0·99	0·72	0·13	−0·34	−0·96	−1·58	−1·90	35·4
	8·6	0·685	1·00	0·71	0·16	−0·26	−0·77	−1·48	−1·73	34·6
	12·6	0·587	1·03	0·66	0·10	−0·30	−0·81	−1·45	−1·72	34·1
	15·6	0·516	0·99	0·80	0·17	−0·30	−0·79	−1·49	−1·81	33·6
May	1·6	0·113	0·89	0·65	0·05	−0·34	−0·86	−1·55	−1·93	30·2
May	4·6	0·036	0·84	0·68	0·02	−0·41	−0·98	−1·58	−1·98	29·6
	5·6	0·010	0·94	0·70	0·05	−0·39	−0·93	−1·64	−1·96	29·2
	6·6	−0·016	0·90	0·63	0·04	−0·44	−1·10	−1·68	−2·06	28·9
	16·6	−0·272	0·78	0·57	−0·05	−0·42	−1·02	−1·63	−1·91	25·3
	29·5	−0·585	0·75	0·67	0·00	−0·46	−1·04	−1·53	−1·85	19·2
May	30·5	−0·607	0·69	0·45	−0·06	−0·47	−1·18	−1·68	−1·96	18·6
June	9·5	−0·811	0·63	0·39	−0·27	−0·69	−1·24	−1·98	−2·20	12·6
	12·5	−0·850	0·57	0·32	−0·31	−0·76	−1·31	−2·01	−2·21	10·4
	24·5	−1·009	0·44	0·22	−0·45	−0·90	−1·40	−2·04	−2·28	1·5
	26·5	−1·028	0·48	0·23	−0·43	−0·84	−1·31	−2·03	−2·19	0·2
June	27·5	−1·034	0·49	0·29	−0·38	−0·81	−1·30	−1·95	−2·14	0·9
July	1·5	−1·042	0·51	0·23	−0·36	−0·70	−1·28	−1·88	−2·05	4·0
	2·5	−1·044	0·56	0·27	−0·33	−0·74	−1·32	−1·86	−2·07	4·8
	3·5	−1·046	0·55	0·32	−0·30	−0·73	−1·24	−1·86	−2·06	5·6
	8·5	−1·045	0·53	0·31	−0·26	−0·64	−1·16	−1·73	−1·96	9·6
July	19·5	−0·973	0·80	0·56	−0·06	−0·56	−1·13	−1·77	−2·07	17·5
	20·5	−0·962	0·78	0·51	−0·15	−0·58	−1·10	−1·74	−1·93	18·2
	22·5	−0·941	0·86	0·58	−0·04	−0·54	−1·12	−1·74	−2·06	19·6
	26·5	−0·898	0·71	0·56	−0·02	−0·49	−1·00	−1·63	−1·89	22·1
	27·5	−0·886	0·87	0·62	+0·03	−0·49	−1·02	−1·62	−1·90	22·6
July	29·5	−0·857	0·86	0·56	0·07	−0·44	−0·95	−1·57	−1·88	23·9
	30·5	−0·843	0·89	0·58	0·02	−0·38	−1·03	−1·65	−1·90	24·4
	31·5	−0·828	0·90	0·52	0·04	−0·45	−0·96	−1·61	−1·88	25·0
Aug.	5·5	−0·756	0·90	0·66	0·04	−0·36	−0·97	−1·58	−1·79	27·7
	26·5	−0·406	1·11	0·85	0·18	−0·29	−0·89	−1·60	−1·88	36·2
Aug.	27·5	−0·389	1·12	0·87	0·20	−0·20	−0·89	−1·53	−1·82	36·6
	28·5	−0·372	1·14	0·81	0·18	−0·26	−0·87	−1·59	−1·84	36·9
	31·5	−0·320	1·18	0·92	0·28	−0·27	−0·89	−1·60	−1·91	37·8
Sept.	15·5	−0·063	1·21	0·93	0·24	−0·24	−0·85	−1·57	−1·71	41·5
	16·5	−0·046	1·24	0·95	0·31	−0·21	−0·78	−1·47	−1·71	41·7
Sept.	18·5	−0·012	1·19	0·98	0·26	−0·24	−0·79	−1·50	−1·67	42·2
	21·5	+0·038	1·25	0·96	0·26	−0·28	−0·84	−1·37	−1·62	42·8
	23·5	0·071	1.22	1.00	0·25	−0·21	−0·82	−1·38	−1·62	43·1
	26·5	0·121	1·24	0·98	0·30	−0·18	−0·79	−1·42	−1·66	43·6
Oct.	20·5	0·506	1·17	0·95	0·29	−0·17	−0·72	−1·45	−1·62	46·7

These residuals are more irregular before opposition than after opposition. Nothing has been detected in the observations and reductions which suggests that the accuracy is in any way greater in the second period than in the first, and the authors are inclined to attribute the greater irregularity to real changes on Mars.

TABLE II

Reduced magnitudes g and coefficients of phase μ

Wave-length A	g_λ 1954	g_λ 1952	μ_λ 1954	μ_λ 1952
4050	+0·43±0·02	+0·14±0·03	0·018±0·001	0·025±0·001
4250	+0·19±0·02	−0·02±0·02	0·017±0·001	0·022±0·001
4550	−0·42±0·02	−0·63±0·02	0·016±0·001	0·021±0·001
4945	−0·82±0·02	−1·14±0·02	0·015±0·000$_5$	0·024±0·001
5430	−1·34±0·02	−1·60±0·03	0·013±0·001	0·021±0·001
5980	−1·96±0·03	−1·98±0·03	0·012±0·001	0·012±0·001
6360	−2·20±0·03	−2·31±0·05	0·011±0·001	0·013±0·001

TABLE III

The Russell–Bond albedo in 1954 and 1952

λ	4050	4250	4550	4945	5430	5980	6360
A_λ 1954	0·038	0·044	0·069	0·087	0·128	0·20	0·24
A_λ 1952	0·037	0·043	0·066	0·078	0·110	0·21	0·24

The residuals have been plotted against the longitude of the central meridian, but the points so obtained do not suggest any correlation which can be relied on.

As the observations secured in 1954 are more numerous than those in 1952, and show greater residuals from the linear phase relation, the authors suppose that the close relation found in 1952 was due to a relatively stable period on the martian surface: and, as a consequence, that not much general information about the physics of the martian surface can be inferred from any short series of observations of magnitude.

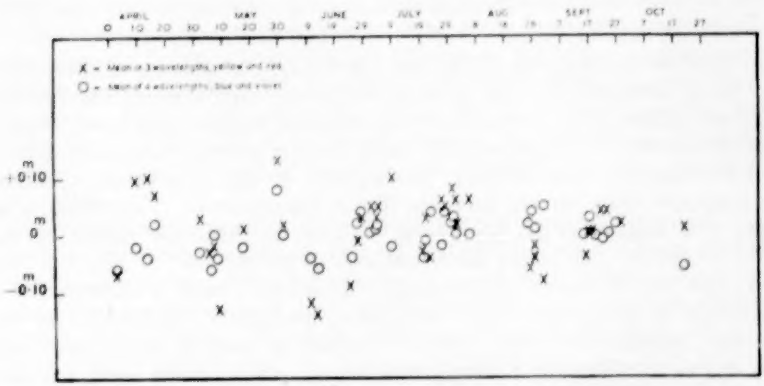

FIG. 1.—*Residuals from* $m = g + \mu\alpha$, *plotted against time.*

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CINEMATOGRAPHY OF PARTIAL SOLAR ECLIPSES

III. Analysis of the observations made at Mombasa, 1948 November 1

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Summary

The observations at Mombasa were the first to be made by this method. The paper describes the method of measuring the photographs, and of determining the zero of P.A.'s; the method of correcting the P.A.'s for the effects of lunar irregularities; the derivation of suitable normal equations; and a number of systematic corrections which should in principle be considered in all eclipse work. The interactions between the Washington limb-traces (which were used to derive the limb-effects) and the eclipse results are discussed, and a direct check on the difference between the mean P.A. system of the two traces used, and that of the eclipse, gave $0^\circ{\cdot}030 \pm 0^\circ{\cdot}018$. The "geodetic" probable errors of the result correspond to ± 34 metres and ± 45 metres in the two directions parallel to the fundamental plane, or to $\pm 0''{\cdot}019$ in $\alpha \cos \delta$ and $\pm 0''{\cdot}025$ in δ, for (roughly) the difference between the Moon's place and the Sun's. The probable error of the Moon's mean longitude is much larger ($\pm 0''{\cdot}064$), owing to the uncertainty in that of the Sun. There are in the present case several known sources of possible systematic error, but they appear quite capable of elimination in future work.

The observational material.—The material brought back from the Mombasa eclipse (**1, 2**) consisted of about 59 metres of 35-mm film, showing the rotation and diurnal motion of the unobscured solar crescent continuously from roughly 96 seconds before mid-eclipse to 76 seconds after (**2**, p. 646, Plate 10), together with a chronograph tape showing the instants of the individual exposures directly against radio signals from the Royal Observatory's Time Station at Abinger (via Rugby, MIK, 9725 kc/s). There was also the record of the latitude, longitude, and height above sea level of the station.

The film was developed commercially; as received back, it consisted of a short stretch of pre-eclipse "sky" exposures, moderately blackened over the frame area, followed by 3 100 "eclipse" frames, on each of which there was a fully exposed crescent on a blank background. Occasionally there was also the black patch near one corner which showed that the "identification-mark" shutter (**2**, p. 641) had been open; this appeared on one frame only on one occasion, on two consecutive frames on twelve occasions, and on longer stretches on two occasions, when this shutter had evidently stuck for periods of roughly a second each. The mark never impinged on the crescent image. The crescent was about 9 mm long; under a magnifying glass, a fair amount of roughness of

the lunar limb was clearly discernible, while the Sun's limb showed no comparable roughness, and since lunar limb-features run only to $\pm 2''$, both the optical definition of the apparatus and the atmospheric seeing must have been good.

The chronograph tape (one pen for the camera shutter and one for the radio signal) showed the start of the run unmistakably, and the radio time marks were clear and were only rarely accompanied by any visible atmospherics. The shutter contacts (about 18 per second) were all well separated except for the brief periods when the camera pen was held over by the "mark" contact which closed whenever the identification-mark shutter was open. This contact had suppressed one "break" on five occasions, two breaks on eight, and longer stretches on two others, evidently in rough agreement with the marks on the film.

Measurement of the chronograph tape.—The tape was measured by Mr C. C. Harris, in the Time Department at Greenwich, according to the standard Observatory procedure. This relates each shutter contact to the two "seconds" contacts which immediately precede and follow it, provided that there is no clear indication that either of these was displaced by an atmospheric. Eight of the 176 time marks used were manifestly displaced (one was advanced by a large amount, and seven were retarded by a total of 125 milliseconds); these were all, in effect, treated as though they had been missing altogether, the interpolation being carried over two seconds (or, in one case, three) instead of one. The possibility that minor atmospherics would more often bias all the signals late than early can never be entirely ruled out when radio time signals are being received; but the net effect on the mean observed "time of mid-eclipse" which the seven visibly late contacts would have produced, if they had been used, would have been less than one millisecond, and it is thought unlikely that there can be any systematic deviation exceeding a very few milliseconds for the *average* of the 168 contacts used.

The two stretches of the camera-pen trace, where the identification signal had been prolonged, were filled by simple arithmetical interpolation, and were found to involve 20 and 13 exposures respectively; the camera ran so uniformly that it would be quite unsatisfactory to suppose that either of these counts was wrong by one, and they also checked with the phase relationships on the 13 other occasions, before and afterwards, when the signal had operated correctly. Fifty-four camera contacts in all had therefore to be supplied by interpolation (mostly of ones and twos), and the loss of weight, out of 3 100 contacts, is negligible.

In 1948, all Greenwich time signals were still of the "rhythmic" pattern, with 61 impulses per minute, and the interpolated shutter-times were thus in "diminished" seconds as they stood. It would have been possible to maintain these units until the final "time" and "rate of change" had been obtained, and then to convert these two values only; but it was preferred to convert earlier. The 3 100 "times" were therefore meaned by tens, and each of these 310 values was converted into true time by adding to the fractional part the integral number of "diminished" seconds elapsed since $04^h 26^m 00^s$, multiplying by 60/61, and then adding $04^h 26^m 00^s$. To these provisional times, the following corrections were then applied:—

(*a*) Correction for signal error. The signal was monitored at Abinger in the normal way, and was treated just as all standard signals are in evaluating the

"final corrections" published regularly by the Time Department. The total correction in this case was $+9$ ms (signal late).

(*b*) Correction for transmission time. This was computed to be $+25$ ms.

(*c*) A computed correction of -7 ms was applied for difference of time-constant as between a circuit consisting of radio output plus chronograph and a circuit consisting of chronograph plus contacts only. (The radio circuit's high impedance makes the time signals record relatively early.)

Corrections for polar variation, for annual variation in the Earth's rotation, and for the difference in light-time as between the station and the Earth's centre, are discussed below (pp. 94–95). The difference between G.M.T. and the mean of the 14 "best" observatories, as published by the Bureau de l'Heure, was zero for November 1.

The 310 normal "observed" times were therefore all increased by 27 ms, and these corrected values were adopted as the independent parameter of the analysis; however, in the final analysis they were all further increased by 56 ms (corresponding to a mis-count of one frame) for the following reason.

When the phase relationship between frames on which a "mark" appeared and contacts which were suppressed on the tape was subjected to a final check, it was found that there was consistently a discrepancy of one frame, compared with what was expected (and what had originally been found). The phase relationships in the camera are very complicated; the mark is not imprinted at the time of exposure, but on a frame that has already moved since the exposure, and with a time lag which is roughly a half-integral number of camera cycles; the re-assessment showed that the difference between the serial number of the first frame on which a mark appears and the serial number of the first contact suppressed on the tape should usually be two but could be one; and similarly (in a slightly different proportion) for the last mark and the last contact, each time that the mark mechanism is operated. However, comparison of the actual film and tape showed that these differences were usually three and sometimes two, but never one. This means that the first contact of all, on the tape, corresponds to a crescent image not found on the film; re-examination of the film verified that this might well be the case. The last of the pre-eclipse "sky" exposures was very heavily over-exposed, indicating that the shutter had stopped in (or very close to) the "open" position; when this happened, the observer would turn the hand-knob until it closed, and if in this particular case he inadvertently turned it backwards instead of forwards the first crescent image would be superimposed on the last "sky" frame and could be lost. It is impossible to discover any trace of a crescent on the frame in question, but the over-exposure is such that this is not surprising. (This point was incorrectly reported in **2**, p. 642; the evidence of the "marks" is however quite conclusive, and the original decision to make provision for such a check has proved to be justified.)

Measurement of the crescent position-angles.—For this purpose a special film-holder had to be constructed; it was designed to position the film so accurately that the direction of its edge could be regarded as constant throughout the entire process of measurement, and need not therefore be measured on every frame. The holder consisted essentially of two spools for the film, about 32 cm apart, with a brass "bridge" leading from one to the other. Along one edge of this bridge there was a fixed steel guide-rail which determined the position of

one edge of the film; the central 5 cm of this rail were cut away, so that the angular control depended on two lengths of film edge about 4 cm long each, with their centres about 9 cm apart. The film was held lightly against these by the pressure of a movable rail, bearing on its other edge and controlled by a light leaf-spring. Over the central region of the bridge, a hole was cut out of the brass plate, and a glass window was let in flush with the brass; the film was held down onto the window by a glass pressure-plate, just less than 35 mm wide, which was pressed down onto it, and prevented from sliding with it, by arms acting at its four corners. A couple of small rollers a few centimetres away held the film down almost to the plane of the window even if the pressure-plate was removed. The flanges of the spools stood just clear of the film, and it was in effect located only by the fixed guide-rail, the pressure-plate, and the extent to which it was paid out from one spool, and taken up by the other; this was done by hand, the spool flanges being 12 cm in diameter.

The entire assembly was rigidly mounted on the turntable of a measuring machine originally designed for measuring sunspots on large-scale solar photographs. The turntable was about 45 cm in diameter, with a good angular scale engraved on silver all round its circumference; the scale could be read, by a pair of fixed verniers, to the nearest 1′ directly. The verniers demonstrated that the centring error, and the random division-errors, were negligible; systematic errors other than the centring error have been assumed to be negligible also. A low-power microscope, with a single fixed fiducial thread (spider-web) in its focal plane, could be traversed above the central region of the circle, riding on stationary precision "ways" which were supported by the base of the instrument; the microscope had to be very gradually traversed during the measurements, as the image worked over from near one edge of the film to near the other.

It had been intended, from the first, to measure the P.A. of the actual line of cusps, and not (as in most previous work) to attempt to locate the centre of the Sun (or Moon) and measure the position-angles of the separate cusps from that point. The procedure was therefore to draw the film along, and slide the microscope as necessary, until a crescent was central in the field of view; rotate the turntable until the line of cusps was almost parallel to the thread; and then by further adjustments in film travel and in orientation to reach a situation in which the extreme tips of the two cusps could both just barely be seen projecting (equally) beyond the thread. The circle was then read. In order to read the film edge, the microscope had to be slid along the bridge until the edge appeared; the edge and the thread were then laid up close together and the circle was adjusted until they were parallel. The edge readings so obtained were very consistent; when the complete film was finally measured (an operation which took a number of days) a couple of edge readings were taken, after two consecutive frames, about once in every 25 frames, and the 248 values so obtained were distributed as follows:—

Circle reading:	162° 18′	162° 19′	162° 20′
No.	55	159	34

The mean is 162°·315; the partial means for successive sixths of the complete run were 162°·313, 162°·314, 162°·316, 162°·315, 162°·317 and 162°·316, and the slight drift thus indicated is small enough to ignore, since the total change in crescent **P.A.** was about 76°. The above scatter included the effects of (*a*) direct

errors of observation, (*b*) departure of the film edge from straightness, (*c*) failure of the edge to be held exactly against the guide rail, and (*d*) any lack of permanence in the position of the film-holder itself on the circle, or of the fixed supports which carried the microscope. It is clear that the film-holder worked reliably, and the mean edge reading constitutes a well-determined datum from which the crescent position-angles can be reckoned.

The zero of position-angles had to be determined with reference to the propulsion holes, not the film edge, and it was therefore necessary to verify that the line joining a pair of holes was perpendicular to the edge; the magnification and field were not such that the full width of the film was visible at one viewing, and the verification was made by special measurements in which the microscope was traversed from one hole to the other, but the effect of the traversing was cancelled by repeating the measurements film down. A fairly long series of tests, on different lengths of the film, demonstrated that the line of holes was indeed perpendicular to the edge to better than 1′, and with a satisfactorily small scatter.

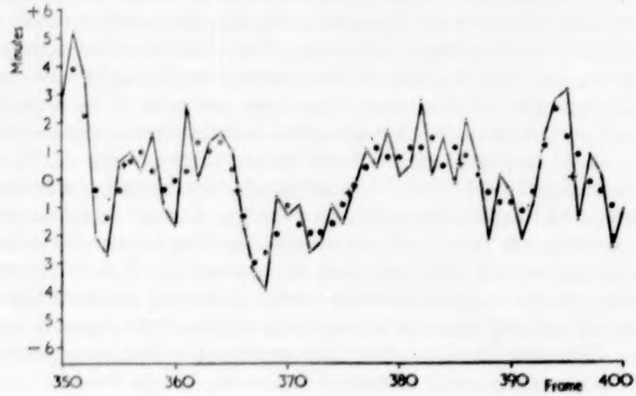

FIG. 1.
• *Individual frames.*
○ *Overlapping means of three.*
+ *Straight means of ten* (*cf. Fig.* 9).

A preliminary survey of the film was then made (by R. d'E. A.) by measuring the crescent position-angle (reckoning from the film edge, i.e. a still undetermined zero) for every tenth frame only, together with measurements of every frame over short selected stretches, from the start of the run down to about frame 2200, when it had to be interrupted. As was expected, the results showed some quite violent fluctuations ($\pm 1^\circ$) from a smooth run, due to the roughness of the lunar limb and the smallness ($3\frac{1}{4}^\circ$ at smallest) of the cusp angle; but in general there was very little doubt about any particular frame. In most cases both cusps were uniquely defined, and the probable error, for repeated re-measurement of one frame, was then about $\pm 1'$; the same value was subsequently obtained by other observers. This meant, of course, that the effects of fairly fine lunar limb-detail might be observable, and some tests were therefore made of the consistency from frame to frame. Fig. 1 shows the actual readings obtained over a suitable

range where they were not disturbed by (visible) cusp irregularities; an arbitrary smooth ephemeris has been removed for convenience. It is evident that the variations from frame to frame are not purely random, and quite fine-grain detail is observable in principle. No actual use was found for this, however, since it is not reproduced in the *E* curve (see below); it may well be that this degree of detail cannot be correlated with limb-traces unless the librations are almost identical in the two cases. The oscillations observed may however also be of non-lunar origin, arising, for example, from slight ripples in the seeing, not in phase at the two cusps.

Not all stretches of the film showed clean-cut cusps; occasionally, one cusp (or even both at once) showed an anomaly, usually lasting for 5 to 15 frames but sometimes up to 50, and changing quite consistently from frame to frame; this took the form either of a reasonably well detached Baily's bead or of a thread-like extension without noticeable neck. In such cases, more than one position-angle could be read, depending on whether the tip of the main cusp or the tip of the appendage was set on; but at least one of the values was still pretty well defined in itself and compromise settings were not made. If one measured every frame, one had sooner or later to make a quite definite change from setting on what had been the cusp, and to transfer to the appendage which had developed beyond it and had grown too strong to ignore; or (at the other cusp) to change from what had started as the cusp, but had gradually become a very faint appendage, and to set on a "main cusp", further in. Such changes naturally involved discontinuous jumps in the run of readings, and a little consideration will show that these jumps must always be "forwards", in the numerical values, no matter which cusp is involved. There is a certain arbitrariness in saying just when a jump ought to be made, but this is paralleled in the "theoretical" limb-effects derived (below) from the known lunar profile.

In addition to the sudden forward jumps in the run of readings, which thus occurred, there were many well-marked (and longer) stretches where the rate of advance was little more than half its overall average for that part of the run; such stretches tended to be rather smooth. These "slow runs" are to be expected: the "leader" cusp will move slowly whenever it is working up any fairly steep slope, and the "follower" will move slowly whenever it is moving down one; in either case the rate of advance, in position-angle, of the line joining the two will be much reduced, and irregularities of the limb at that cusp where the slope is steep will also have less effect than usual. When a smooth ephemeris is removed, the residuals thus show a "sawtooth" pattern, which may be considered typical of this method. (Figs. 8 and 9; the points plotted are now means of ten.) Neither these sawtooth features nor any others, in the run of the observed position-angles, bear any superficial resemblance whatever to the lunar profile which causes them.

A number of tests were made to estimate the magnitude of possible systematic errors. Since the two cusps diverged by about 30°, any residual astigmatism in the observer's eye might perhaps tend to weaken one cusp more than the other; and small differences in P.A. might indeed be expected between different observers, and between the two eyes of one observer, whether a clear reason could be given or not. The fiducial wire was necessarily kept fixed, and the orientation of the crescent thus appeared the same in the eyepiece throughout the entire run, so that any such effect might be systematic. Differences were

indeed found, both between the two eyes of one observer and between different observers, amounting to 1′, or even 2′, and such differences are rather too large to ignore if the internal probable error of the final results is the criterion; in the present instance, however, they have been ignored, since there are other systematic errors which may be larger; the main purpose of this first attempt was to establish the general serviceability of the method, especially as regards the closeness with which the measurements would agree with what the lunar profile demanded, i.e. the smallness of the internal probable errors. The most satisfactory way of studying, and eliminating, such systematic errors, in future cases, without increasing the large amount of measuring that is already necessary, would probably be by incorporating a reversing prism in the eyepiece, and rotating it (say) 18° between each frame and the next, so that only every tenth image appears in any one orientation; there might also be some advantage in measuring all the odd frames film up, and then all the even ones film down, which would involve no real increase in total labour.

Systematic error may also be expected when one cusp is blunter than the other; this, however, should not occur, at least with a good optical system, except temporarily, and then with equal probability at either cusp. It thus becomes essentially a random error, though of relatively long period.

A considerable amount of time was devoted to studying these matters. It was finally decided that for the purpose of this first attempt it would be sufficient to measure the entire film straight through, in one arrangement only. It seemed improbable that the total systematic error so introduced could reach 0°·05, and the uncertainty which appears (in the present instance) to have affected the zero of position-angles was thought at the time to be larger.

On this basis, the systematic measurement of the entire film was then entrusted to Mr E. A. Whitaker, of the Department of Astrometry. A good deal of effort and strain can be saved if one is content with a probable error of possibly 2′ instead of 1′, and the large number of frames was considered to justify some relaxation; even so, however, the work took many hours. A direct comparison of his figures with those obtained by R. d'E. A. in his preliminary survey, for 208 frames measured by both and not marked as showing some appendage, gave a mean difference of 1′·4 with a formal probable error (per single frame) of ±2′·9; the differences actually showed long-period systematic oscillations, but this probable error has been computed as though they were random. Any variations could almost certainly be made nearly random, in future work, by measuring the odd frames in the "direct" order and the even ones in the "reverse"; it seems likely, however, that a reversing eyepiece would in any case eliminate most sources of personal error.

The 3100 crescent P.A. values so obtained were meaned by tens, working formally to thousandths of a degree. (In 27 cases in the last thousand frames the image appeared double or poor and was not measured; a value was supplied by rough interpolation, taking account of the general run of the ten and the position within it of the missing one.) The true position-angles increased during the run, while the readings of the measuring machine decreased; the 310 values had thus to be subtracted from the mean reading for the film edge (162°·315), and the results converted to true position-angles by adding the position-angle of the direction in the sky which corresponded to the film edge, together with certain small corrections, which will be described below.

Determination of the zero of position-angles.—It was intended to determine the zero of position-angles by comparing the direction of the line (nearly E–W) joining a pair of propulsion holes with the direction of the diurnal motion as inferred from the common tangent to all the 3 100 south limbs of the Sun, the south limb being unobscured throughout the run. For this purpose the complete film-holder was transferred to a one-screw plate-measuring machine ordinarily used for astrometric plates, and was set with the film edge (i.e. approximately the N–S direction) nearly parallel to the screw. By suitably advancing the film one could bring a crescent into the field of view and place it with its south limb close to the measuring wire; and by traversing in the crossways of the machine (i.e. parallel to the wire) one could then bring either one of the appropriate pair of holes into view instead, keeping the film stationary in the holder. It was thus possible to take three readings for each frame; on the limb ($\odot$), on the appropriate "east" hole (E), and on the corresponding "west" hole (W); and so to form for each frame the "north–south" coordinate of the south limb with respect to the holes, $\frac{1}{2}(E+W)-\odot=G$, say. The machine read to microns, and the edge of a hole, being straight and parallel to the measuring wire, could be set on with an uncertainty of a very few microns. The east–west coordinate could also be measured roughly; its total travel was about 11·2 mm. For actual use, however, the length travelled was computed from the focal length (935 mm), the declination, and the total time; the value so obtained was 11·38 mm.

The 3 100 values of G were not all worked out individually; the values of $\odot$, E, and W were summed by pages (40's) and a mean value of $\frac{1}{2}(E+W)-\odot$ for each page was obtained directly from these sums. However, a general check on the individual values was readily available, since the measuring procedure was to set the micrometer back to a standard figure after each measurement of W, pull the film along until the limb of the next crescent was very close to the wire, and then measure $\odot$, E, and W in that order; it followed that the readings for $\odot$ were always close to the standard one, and any large anomalies in G could be picked up by inspection, in both the E and W columns. A complete page of individual G-values was computed occasionally, and half the values ordinarily showed residuals, from the mean for the page, numerically less than about 9 to 12 microns; the distribution was not normal (usually showing a rather sharp cut-off), but the scatter can be qualitatively described by a "probable error" of $\pm 12\,\mu$ or less. All residuals greater than $\pm 30\,\mu$ were marked, with an additional mark when they exceeded $\pm 70\,\mu$; in the first 1653 frames there were only 33 of the former class and none at all of the latter. Indeed, there were none even of the former from frame 745 to frame 1183, and only one of each sign in the further stretch from frame 1208 to frame 1653, and there can be little doubt that the consistency shown by these 900 frames represents the performance which the camera is designed to give. Even if fiducial marks were not introduced in future work, this would be a sufficiently small scatter if it were purely random; it would lead to an uncertainty of a small fraction of a minute, in the zero of position-angles.

Unfortunately, this consistency did not continue in the second half of the run. Frame 1654 showed a residual of $-138\,\mu$, and occasional negative residuals lying between $-100\,\mu$ and $-140\,\mu$ occurred on most pages from that point on. In general, the number of residuals (of either sign) which exceeded $30\,\mu$ but did

not reach $70\,\mu$ remained small, with no large positive residuals; the conclusion appears inescapable that large negative residuals represent an anomaly, liable to occur on individual frames but not disturbing the general distribution in the other cases, and that the proper procedure is simply to exclude the anomalous frames and accept the resulting loss of weight.

There seems no doubt that in the anomalous cases the film had failed to pull quite as completely into place, in the camera, as it did in the normal ones. At the end of each "pull" it was clamped by the pressure-plate, with one hole forced down on to a fixed peg; but it was now found that there was a play of over $\frac{1}{10}$ mm on this peg if the pressure-plate was released, and it seems clear that the film was always near one or other of the two possible limiting positions, and in the first half of the run always near one of them. The number of anomalous frames became seriously large towards the end of the run, but it still appeared that the proper treatment was simply to exclude them; for example, the page of frames 3021–3060 showed six negative residuals exceeding $70\,\mu$, reckoning from the original mean of the page, and it also showed four positive ones $> +70\,\mu$; but the fact that two compact groups (now nearly equal) were really involved was made clear by the circumstance that only one of the entire 40 residuals now fell in the range between $-30\,\mu$ and $+30\,\mu$. Thus even in this extreme case it was possible to distinguish with certainty between "normal" and "anomalous" frames; when they had been sorted out, the "normal" group comprised 21 of the original 40, with only one residual (from their mean) numerically greater than $30\,\mu$ ($-35\,\mu$), and with half of them numerically less than $11{\cdot}5\,\mu$ (i.e. with an acceptable scatter), while the anomalous 19 frames had a mean value of $119\,\mu$ smaller, and formed a group fully as compact as the normal one, the extreme range of the residuals (from the mean of this group) being from $-27{\cdot}3\,\mu$ to $+30{\cdot}2\,\mu$, with half of them numerically less than $13{\cdot}3\,\mu$. The gap between the largest value in the low group and the smallest in the high one was $53\,\mu$, and there is nothing arbitrary in the criterion for separating them. The diurnal run should thus be determinable if the "normal" frames only are used, the "anomalous" ones being all rejected. (In principle, one might also use only the "anomalous" ones; the result would be identical if their mean difference was constant, but it would have very much less weight. The mean difference was not actually evaluated; if it had been, and if it had proved constant, it could have been applied to all the anomalous values, but the labour would have been considerable and the gain in weight rather small.)

The mean values of G are plotted in Fig. 2 against frame number; both the uncorrected means are given and also the "corrected" ones, obtained by simply dropping the anomalous frames. It is at once apparent that the corrected points not only scatter much less than the uncorrected ones, but also follow on considerably better from the first half of the run. The fact that the Sun's declination was not zero means that there should be a slight concavity upwards, and the corrected points do also indicate this, though too strongly; the uncorrected ones forbid it altogether.

When the "corrected" points are re-plotted with a smooth ephemeris of $-11\,\mu$ per 400 frames removed (Fig. 3), it is seen that the general appearance of the run really suggests two different straight lines. It is certain that there cannot have been any sudden sideways tilt of the camera bench, rotating the film in its own plane by $0^{\circ}{\cdot}4$ or thereabouts in a short interval of time near

G

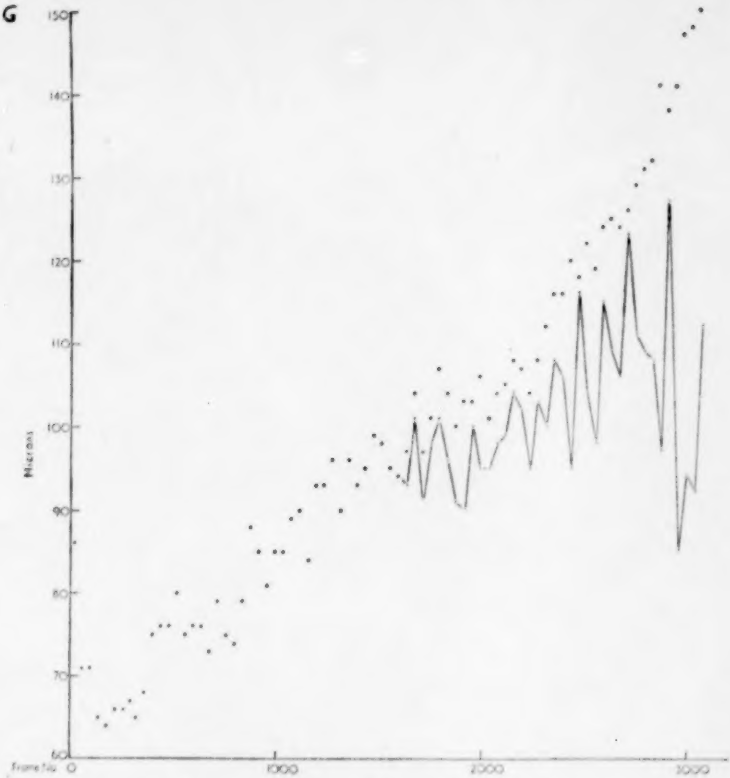

FIG. 2.—*Zero of position-angles.*
. — . *Uncorrected points.*
. . . *Corrected points.*

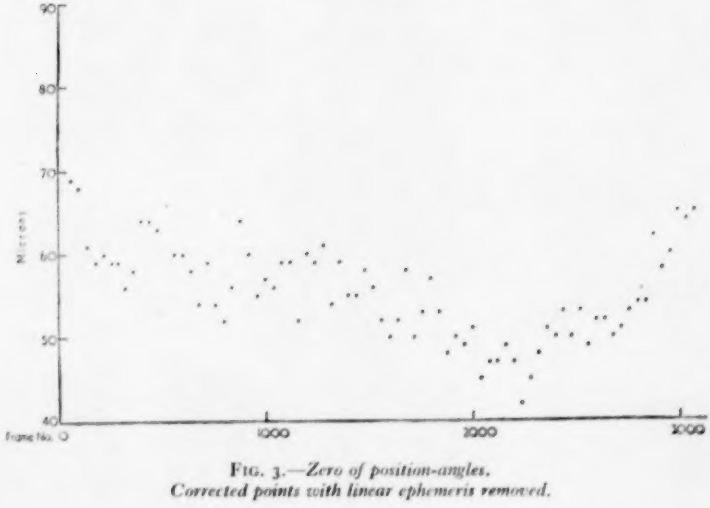

FIG. 3.—*Zero of position-angles.*
Corrected points with linear ephemeris removed.

frame 2200. It is not merely that since the light was first reflected by an oblique mirror, carried on the same bench, such a twist would be expected to produce very large discontinuities in the absolute values of both coordinates in Fig. 3, as well as in the slope; even if these discontinuities happened both to be zero, a real rotation of the camera would necessarily introduce a discontinuity of $0^\circ{\cdot}4$ into the measured position-angles themselves and, as will be seen below, there is evidently no such discontinuity in these values at all.

In the early stages of the work, since there seemed no good reason to favour one part of the diurnal run at the expense of the other, a straight line was drawn (by eye) through the entire array of "corrected" points in Fig. 2; its slope was $+75\,\mu$ in 3100 frames. The computed length of the diurnal run being 11·38 mm, the slope of the "common tangent" so drawn was $0^\circ{\cdot}378$, but with serious uncertainty. However, at a late stage of the work several independent considerations strongly suggested (p. 85) that it might definitely be the final stretch of the run which depicted the true diurnal direction, some disturbing factor having been operative until frame 2200 or thereabouts. A discontinuity in slope, with none in the actual ordinates, could result from a slow drift in the longitudinal tilt of the whole bench (or possibly of the prism only), which persisted at a nearly constant rate until about frame 2200 and then definitely stopped; a drift totalling about $10''$ (instead of $0^\circ{\cdot}4$) would now be adequate, and such a small drift cannot be certainly excluded in the Mombasa case.

It was decided, therefore, to adopt the run of the last 900 frames exclusively, for determining the zero of position-angles, and to evaluate this slope rigorously, since the small scatter here permits this. For this purpose, the abscissa scale was changed from frame numbers (though the camera did in fact run with great uniformity) to the stricter

$$X = \frac{15 f \cos\delta}{3438}(t - t_0), \tag{1}$$

where f is the focal length (935 mm) and $t - t_0$ the time from (computed) mid-eclipse in minutes. The ordinates G were also diminished by the computed effect, $\Delta G = X^2/(2f\cot\delta)$, of the curvature of the diurnal circle, so that the slope refers strictly to the moment of mid-eclipse. A least-squares solution was then made for the 23 normal points representing the means of 40-frame stretches centred on 2200, 2240, ... 3080, with approximate weights when the totals departed appreciably from 40 owing to the omission of anomalous ("A") frames. The result was

$$G - G_0 = (0^\circ{\cdot}683 \pm 0^\circ{\cdot}022)X \tag{2}$$

and the value $0^\circ{\cdot}683$ was adopted for the final work. In principle, it is still subject to an error caused by differential refraction, but the diurnal track was almost vertical in the present case, and the computed error was only $0^\circ{\cdot}002$ and was ignored. The difference between this system and all the earlier ones is about $0^\circ{\cdot}3$ in position-angle, which is equivalent to about 0·6 sec change in t_0, or $0''{\cdot}3$ in the Moon's place; the systems thus differ very substantially.

It is evident that the above probable error does not really represent the reliability of the figure, in the present case, but the weakness should not recur in future work. Not only have fiducial marks since been arranged in the cameras, so that the difference between "anomalous" and "normal" frames effectively disappears; the horizontal position in which the bench can generally be used also allows it to be properly clamped down onto flat metal bedplates,

and with a good concrete pier a much higher stability should then be possible than could be guaranteed in the Mombasa case. It seems probable that the zero of position-angles should in future be obtainable with a formal probable error well below $\pm 0^\circ{\cdot}01$, and with good reason to hope that this would represent the true uncertainty.

The fact that the film sometimes failed to pull completely into place, in the camera, now suggested that it might also take up an anomalous direction at the end of the pull; the position-angle of an image, measured from the direction of the film edge, would then be anomalous as well as the actual position itself. If this happened at random, it might be hard to detect, though it would certainly cause a scatter; but the question whether it happened preferentially with the frames already noted as anomalous is easily investigated. For this purpose, all the frames on which the value of G had been anomalous, in the work on the zero, were now marked "A" in the record of the position-angle readings themselves, and the circle reading for each "A" frame (already recorded) was compared with the mean of the preceding and following "regular" ones. Where two or more "A" frames were consecutive, their mean was compared with the mean of a correspondingly increased number of adjacent "regular" ones, and where two or more "A" frames alternated with regular ones the mean of the odd ones was compared with the mean of the even ones; in all cases the differences were given a weight equal to the number of "A" frames involved. On the last three pages (frames 2950–3100) there were so many "A" frames that these procedures were unsuitable; the last 150 P.A. readings were therefore all plotted, smooth curves were drawn through the "A" points and the regular ones separately, and the mean differences between the two curves were read off. (These curves are necessarily rather irregular, since they contain the full effect of the lunar limb detail.) The systematic differences, in the sense circle reading "A" *minus* "regular", were thus found to be as follows:—

frame	1654 to 2150	2151 to 2950	2951 to 3000	3001 to 3050	3051 to 3100
difference	$+5'{\cdot}0$	$+6'{\cdot}5$	$+7'{\cdot}2$	$+5'{\cdot}0$	$+5'{\cdot}8$

The variations in the values from frame 2951 on are barely significant, and a mean value of $+6'{\cdot}0$ is sufficiently near for actual use; even when there are five "A" frames in one set of ten (which happened only towards the very end), the difference between $5'$ and $6'$ is a matter of only half a minute in that one of the 310 normal points. It was therefore decided simply to decrease the value for any given mean of ten circle readings by $0^\circ{\cdot}010g$, if there were g "A" frames in that ten, throughout the entire range affected by anomalies; in what follows, the values of P' are the 310 means of ten circle readings, as so corrected. (It was found that most of the 27 cases in which the image had been too poor to measure were "A" cases; as already explained, values for these had been supplied when the means were first formed, and since these supplied values were as a rule little affected by the systematic difference now discovered, they were not corrected for it.)

It was verified by inspection of the camera mechanism that the sign of the correction is what one would expect if the error arose through forcing the film down onto the positioning peg; it seems reasonably certain that the film would lie in the "normal" direction whenever the following edge of the hole was

pressed against the peg, and in an abnormal direction when the leading edge of the hole was, so that it really would only be in those cases in which the position of the image was anomalous that one would expect its position-angle to be so. These disturbances will of course also disappear in any camera fitted with fiducial marks.

The angle $0^\circ{\cdot}683$ obtained above (eqn. (2)) represents strictly the extent to which the "mid-eclipse" tangent to the slightly curved diurnal run failed to be parallel to the cross-ways of the measuring machine, as the film was actually mounted in the machine. It was mounted with the line joining an "E" hole to a "W" one very nearly parallel to the cross-ways, but not exactly; on the average, the value of the difference $E - W$ was about $+5{\cdot}3\,\mu$. (The scatter from this showed some small systematic features, but these have been ignored.) Since the image approached the W edge as the run proceeded, and since the holes are 28 mm apart, the angle $0^\circ{\cdot}683$ is larger by $57^\circ{\cdot}3 \times 5{\cdot}3/28000 = 0^\circ{\cdot}011$ than it would have been if the film had been mounted quite correctly; the tangent therefore failed to be perpendicular to the film edge by $0^\circ{\cdot}672$. (A little consideration shows that lack of perpendicularity between the two "ways" of the measuring machine does not enter as well.) Since increasing readings corresponded, on any one frame, to increasingly north coordinates, the holes ran increasingly north of the Sun as the eclipse proceeded.

The diurnal run was not, however, due west, owing to the change in the Sun's declination. The "diurnal" slant Y is $\dfrac{\sec\delta \,.\, \Delta\delta}{15\,.\,\Delta t}$ (radians) if $\Delta\delta$ is the change in δ (in seconds of arc, say) in the time Δt (in seconds), and δ is the declination of the Sun's centre. (The point set on was always a constant distance from the Sun's centre.) We thus have

$$Y = -\frac{1{\cdot}0324 \times 1156''{\cdot}4 \times 57^\circ{\cdot}3}{15'' \times 86400} = -0^\circ{\cdot}0528.$$

The diurnal track ran south of true west, while the line of holes ran north of the track, so that the line of holes ran $0^\circ{\cdot}672 - 0^\circ{\cdot}053 = 0^\circ{\cdot}619$ north of true west, or in P.A. $270^\circ{\cdot}619$. (See Fig. 4.) Accordingly, the P.A. of the

FIG. 4.

film edge, as judged by the tangent to the diurnal motion at mid-eclipse, was $0^\circ{\cdot}619$, and the figure from which the (means of) circle-readings, P', have to be subtracted, in order to convert them to position-angles of the line of cusps, is $162^\circ{\cdot}315 + 0^\circ{\cdot}619 = 162^\circ{\cdot}934$.

The absolute "observed" position-angles.—So far we have assumed that there are no systematic falsifications of the P.A. due to lens distortion or to projection effects. The lens was tested for distortion by strapping it to the Astrographic mounting and using it to photograph the Pleiades. This was done through a Wratten 58 (green) filter, to parallel the eclipse conditions, and

the images were "extremely good" all over the field ($1\frac{1}{2}°$). The computed standard coordinates were compared (in the Department of Astrometry) with the measured places for a central region, for an intermediate zone, and for the outer region of the field separately, and no detectable difference of scale was found; it was therefore concluded that distortion was too small to have any influence on the results.

The lens thus gives a simple gnomonic projection, with all hour-circles projecting strictly as straight lines; within the small field used, these diverge at $15' \sin\delta$ per minute of time. It is readily verified that over this field, in view of the fact that projection errors do not affect the position-angle when the two cusps are either in line with the centre of the field or both equidistant from it, it is sufficiently accurate to apply the above rate of rotation to the entire field; projection errors can be neglected in all cases. Accordingly, the apparent position-angles differ from $162°{\cdot}934 - P'$ by

$$\Delta P = 15' \sin\delta\,(t - t_0) = -0°{\cdot}0622(t - t_0). \qquad (3)$$

It is not necessary (as one might imagine) to correct for the fact that the centre of the line of cusps runs ahead of, or behind, the Sun's centre by an amount (nearly) proportional to $\sin M \sec(M - M_0)$, where M is the P.A. of the line of centres; the measurements can be considered as though they referred to a line through the Sun's centre truly parallel to the line of cusps, and since it is convenient to compute M at the Sun's centre it is strictly correct also to use the rotation of field appropriate to that point.

Since the Sun's declination was negative, the convergence makes the position-angles at the start of the eclipse appear larger than they should, and at the end smaller, so that ΔP must be subtracted from the measured values, these having been obtained with reference to a fixed direction. It is important to include this correction for "rotation of the field", however uncertain the absolute zero of position-angles might be, for this term goes straight into the observed rate of change of $(M - M_0)$, (eqn. (4), etc.) and the rate of change is well observed even if there is no good zero at all. The coordinate of the Moon's place perpendicular to its direction of apparent relative motion is inferred from this rate of change, and both the present term and the one discussed in the following section must be strictly applied in all cases.

These apparent position-angles have now to be converted to true "observed" position-angles by removing the effect of differential refraction. This should strictly be computed for the actual P.A. of the line of cusps, as affected (sometimes to the extent of a degree) by lunar limb detail, but it is sufficient to take the P.A. for a smooth Moon, and amply sufficient to use for computing this the formula

$$\tan(M - M_0) = k(t - t_0) \qquad (4)$$

with preliminary values of k and of the mid-eclipse constants M_0 and t_0 (**1**, p. 28, eqn. 9). Since M is the P.A. of the line of centres, and Q (the parallactic angle) is the P.A. of the vertical, the inclination of the line of cusps to the horizontal is $M - Q$, and the correction R', which naturally vanishes when $M - Q = 0$ or $\pm 90°$, can be computed in the usual way. The actual values are summarized in Table I.

TABLE I

$t - t_0$	$-1^m{\cdot}5$	$-1^m{\cdot}0$	$-0^m{\cdot}5$	$0^m{\cdot}0$	$+0^m{\cdot}5$	$+1^m{\cdot}0$	$+1^m{\cdot}5$
R'	$+0°{\cdot}010$	$-0°{\cdot}009$	$-0°{\cdot}031$	$-0°{\cdot}046$	$-0°{\cdot}049$	$-0°{\cdot}040$	$-0°{\cdot}028$

The final expression for the observed position-angles of the line of cusps is then

$$P'' = 162^\circ{\cdot}934 + 0^\circ{\cdot}0622(t - t_0) + R' - P'$$

or, for the line of centres (as falsified, still, by limb-effects),

$$P = 252^\circ{\cdot}934 + 0^\circ{\cdot}0622(t - t_0) + R' - P', \tag{5}$$

where P' is the mean of ten circle readings, corrected for "anomalous" frames (where necessary) as already described. If M is the computed position-angle of the line of centres, the quantities $P - M$ may thus be called the "observed limb-effects" in P.A.

When suitable lunar profiles are available, the "theoretical limb-effects", E, can be computed, and the residuals $P - M - E$ will form the basis of any least-squares solution for the Moon's place; it is useful to split them up for graphical study, and to plot separately a curve of $P - M$ values and a curve of E values. (The final curves will be seen below, Fig. 9, p. 100.) If the available limb profiles are exactly suitable, and if there is no error in the observations, these two curves should be identical in shape; and whether or not they are identical they can be adjusted to have the same mean height and mean slope (both will be nearly zero) by adjusting the Moon's place, since this changes the mean height and mean slope of the function M. (Alternatively, of course, one might adjust the coordinates of the station, or the Sun's place.)

The necessary equation of condition for this adjustment was obtained for only two variables, which can be taken as the corrections to the two coordinates of the Moon's place, and only their effect on M was allowed for. They have also some small effects on E, but it is inconvenient to include these explicitly and they are best dealt with separately. The further analysis thus falls into three main parts: (i) the computation of the M-values, i.e. of the P.A. of the line of centres, for the 310 absolute times involved, using some assumed place (the Almanac one, or a later approximation) for the Moon; (ii) the computation of the 310 limb-effects, E, which involves computing crescent lengths and cusp angles, and measuring the appropriate (Washington) limb-traces; and (iii) the formation of the equation of condition, and of the normal equations.

The computed position-angles.—The approximate formula (4) gives values of $M - M_0$ which are good, but not quite sufficient for the observational accuracy reached, and a rigorous formula is necessary. In the standard Besselian coordinate system the observer is at the point (ξ, η, ζ) where

$$\left.\begin{aligned} \xi &= \rho\cos\phi' \,.\, \sin(\mu - \lambda), \\ \eta &= -\rho\cos\phi' \,.\, \cos(\mu - \lambda) \,.\, \sin d + \rho\sin\phi' \,.\, \cos d, \\ \zeta &= \rho\cos\phi' \,.\, \cos(\mu - \lambda) \,.\, \cos d + \rho\sin\phi' \,.\, \sin d. \end{aligned}\right\} \tag{6}$$

ρ and ϕ' are here the geocentric distance (in units of the Earth's equatorial radius) and geocentric latitude of the observer, deduced in the usual way from the true latitude ϕ, the assumed ellipticity $(\frac{1}{297})$, and the height of the station; λ is the longitude of the observer; μ and d are the Greenwich hour angle and declination of the point on the celestial sphere towards which the axis of the shadow is instantaneously directed; and the "fundamental plane", $\zeta = 0$, is perpendicular to that axis. μ, $\sin d$, and $\cos d$ are tabulated in the Almanac, at ten-minute intervals; $\ddot{\mu}$ and $\ddot{d}$ are completely negligible.

If $\dot{\mu}$ and $\dot{d}$ denote the rates of change of μ and d in *radians* per minute, we have, neglecting $\dot{d}^2$, $\ddot{\mu}$ and $\ddot{d}$,

$$\left.\begin{array}{lll} \dot{\xi}=\dot{\mu}(\zeta\cos d-\eta\sin d), & \ddot{\xi}=-\dot{\mu}^2\xi, & \dddot{\xi}=\text{etc.} \\ \dot{\eta}=\dot{\mu}\xi\sin d-\dot{d}\zeta, & \ddot{\eta}=\dot{\mu}(\dot{\xi}\sin d+2\dot{d}\xi\cos d), & \dddot{\eta}=\text{etc.} \\ \dot{\zeta}=-\dot{\mu}\xi\cos d+\dot{d}\eta, & \ddot{\zeta}=\dot{\mu}(-\dot{\xi}\cos d+2\dot{d}\xi\sin d), & \dddot{\zeta}=\text{etc.} \end{array}\right\} \quad (7)$$

where $\dot{\xi}$, $\ddot{\xi}$, $\dddot{\xi}$, etc. are first, second, and third derivatives with respect to the time (in minutes). We are thus able to express ξ, η, ζ as polynomial functions of the time from any desired zero, preferably an epoch, t_1 say, fairly near that of mid-eclipse. As a check we may use

$$\xi^2+\eta^2+\zeta^2=\rho^2 \quad \text{and} \quad \xi\dot{\xi}+\eta\dot{\eta}+\zeta\dot{\zeta}=0,$$

both of which are of course valid at all times.

The coordinates of the axis of the shadow on the fundamental plane, (x, y), are tabulated in the ephemerides for every ten minutes, and are thus also readily expressed as polynomial functions of the time, measured from the same epoch t_1 as the expressions for ξ and η; we thus have polynomials for $x-\xi$ and $y-\eta$, reckoned from this origin.

In the standard notation

$$\left.\begin{array}{ll} x-\xi=m\sin M, & \dot{x}-\dot{\xi}=n\sin N, \\ y-\eta=m\cos M, & \dot{y}-\dot{\eta}=n\cos N, \end{array}\right\} \quad (8)$$

where (m, M) is the vector (perpendicular distance in Earth-radii, and direction from north), from the observer to the axis of the shadow, and (n, N) is the vector rate of change (Earth-radii per minute, and angle from north) of (m, M). M is also the position-angle of the line of centres of Sun and Moon, as seen by the observer, and $m(\pi_{☾}-\pi_{☉})$ is their apparent separation. We may extend this notation, and write

$$\left.\begin{array}{ll} \ddot{x}-\ddot{\xi}=r\sin R, & \dddot{x}-\dddot{\xi}=s\sin S, \\ \ddot{y}-\ddot{\eta}=r\cos R, & \dddot{y}-\dddot{\eta}=s\cos S, \end{array}\right\} \quad (9)$$

and taking the angles M, N, etc. to be in radians, the polynomials for $x-\xi$ and $y-\eta$ then appear in the convenient form

$$\begin{aligned} y-\eta+\mathrm{i}(x-\xi) &= m\,\mathrm{e}^{\mathrm{i}M} \\ &= m_1\mathrm{e}^{\mathrm{i}M_1}+n_1\mathrm{e}^{\mathrm{i}N_1}(t-t_1)+\tfrac{1}{2}r_1\mathrm{e}^{\mathrm{i}R_1}(t-t_1)^2 \\ &\quad +\tfrac{1}{6}s_1\mathrm{e}^{\mathrm{i}S_1}(t-t_1)^3+\ldots, \end{aligned} \quad (10)$$

where the suffix 1 indicates that the corresponding differential coefficients (etc.) have been evaluated at time t_1.

The time t_1 is arbitrary, and a similar formula can of course be written down for t_0, the estimated time of mid-eclipse. Since mid-eclipse is defined by $N_0-M_0=\pm\pi/2$ (indicating that m has reached its minimum value), we can eliminate N_0 and write

$$m\,\mathrm{e}^{\mathrm{i}M}=m_0\mathrm{e}^{\mathrm{i}M_0}\pm in_0\mathrm{e}^{\mathrm{i}M_0}(t-t_0)+\tfrac{1}{2}r_0\mathrm{e}^{\mathrm{i}R_0}(t-t_0)^2+\tfrac{1}{6}s_0\mathrm{e}^{\mathrm{i}S_0}(t-t_0)^3\ldots, \quad (11)$$

where the suffix 0 indicates that the corresponding differential coefficients have now been evaluated at time t_0; and, multiplying through by $\mathrm{e}^{-\mathrm{i}M_0}$,

$$\begin{aligned} m\,\mathrm{e}^{\mathrm{i}(M-M_0)} &= m_0\pm \mathrm{i}n_0(t-t_0)+\tfrac{1}{2}r_0\,\mathrm{e}^{\mathrm{i}(R_0-M_0)}(t-t_0)^2 \\ &\quad +\tfrac{1}{6}s_0\,\mathrm{e}^{\mathrm{i}(S_0-M_0)}(t-t_0)^3\ldots \end{aligned} \quad (12)$$

Separating this into real and imaginary parts, and dividing the latter by the former, we have

$$\tan(M-M_0) = \pm \frac{n_0}{m_0}(t-t_0) + \tfrac{1}{2}\frac{r_0}{m_0}\sin(R_0-M_0)(t-t_0)^2$$
$$\mp \tfrac{1}{2}\frac{r_0}{m_0}\frac{n_0}{m_0}\cos(R_0-M_0)(t-t_0)^3 + \ldots. \qquad (13)$$

The convergence of this series is not immediately evident, since $|t-t_0|$ can be as large as $1^{m}{\cdot}6$; however, subsequent coefficients are all of the order s_0/m_0 or of the order $\left(\frac{r_0}{m_0}\right)^2$ or of still smaller ones, and in the present case we are dealing with approximately the following values:—

$$\left.\begin{array}{ll} m_0 \simeq {\cdot}0158; & r_0 \simeq 1{\cdot}8\times 10^{-5}; \\ n_0 \simeq {\cdot}0086; & s_0 < 4\times 10^{-8}; \end{array}\right\} \qquad (14)$$

so that $(r_0/m_0)^2 \simeq 10^{-6}$ and $s_0/m_0 \simeq 2\times 10^{-6}$. Since $\frac{1}{6}(t-t_0)^3$ is less than 1, and $\frac{1}{4}(t-t_0)^4$ less than 2, we may neglect all further terms, whatever the phase-angles R_0 and S_0, and still be sure that $\tan(M-M_0)$ is correct to five places of decimals. Since, in addition, $\sec^2(M-M_0)$ becomes appreciable whenever $|t-t_0|$ is large, $M-M_0$ is guaranteed to half a thousandth of a degree at all times; and as the thousandths are only required for rounding-off and checking, formula (13) is as it stands the (sufficiently) rigorous substitute for (4).

The upper signs in (11), (12), and (13), which correspond to $N_0-M_0 = +\pi/2$, must be used when the station is south of the centre-line (as it was in the present case), and the lower signs must similarly be used when it is north; except for some rare and unimportant cases where $|\mu-\lambda| > \pi/2$.

To determine the actual values of M we require also a knowledge of M_0 and of t_0; these are of course obtained directly from the polynomials in $x-\xi$ and $y-\eta$, essentially in the usual manner.

The computed effect of limb irregularities.—It was fortunate for this work that the investigations of Dr C. B. Watts, of the U.S. Naval Observatory, had reached a fairly advanced stage; he had a large number of limb-traces available, and his kindness in selecting and lending suitable ones is very gratefully acknowledged. The limb-traces produced by his machine are of a quality (and quantity) immensely superior to anything available in work on earlier eclipses; important and valuable as the information is which can be obtained from Hayn's classical "Selenographische Koordinaten", the present work gains greatly from being able to substitute the large-scale and very detailed traces obtained by Watts, and especially also from being able to use his "revised" base-lines (see below, p. 84).

The ultimate aim of Watts' work is to produce a collection of lunar data from which the limb-contour for any desired pair (l, b) of libration coordinates can be evaluated; this stage has not yet been completed, but his original traces afford more complete and detailed information, for the particular librations at which they were actually obtained, than could easily be compressed into a general "library" of data of manageable size at all; naturally, he will have no traces exactly corresponding to any arbitrary (l, b) pair that may be required, but he has a very large number of traces altogether, and several came close to having the right librations for the present case. The geocentric librations for the time of mid-eclipse (1948 November 1·186) were $l_0 = +4^\circ{\cdot}12$, $b_0 = +0^\circ{\cdot}38$, and the

differential corrections for Mombasa, computed by Atkinson's formulae (3), were $\Delta l = +0^\circ{\cdot}89$, $\Delta b = -0^\circ{\cdot}26$, so that the topocentric values were $l = +5^\circ{\cdot}01$, $b = +0^\circ{\cdot}12$. Dr Watts was able to supply traces as follows:—

No.	43	94	275/6	J65007/8	J65081/2
Date	1948 June 17·1	1949 Feb. 17·4	1950 Aug. 2·4	1950 Aug. 30·0	1950 Sept. 25·9
Limb	West	East	East	East	West
l	$+5^\circ{\cdot}23$	$+5^\circ{\cdot}07$	$+5^\circ{\cdot}04$	$+4^\circ{\cdot}68$	$+4^\circ{\cdot}80$
b	$-0^\circ{\cdot}52$	$+0^\circ{\cdot}27$	$+0^\circ{\cdot}48$	$-1^\circ{\cdot}04$	$-0^\circ{\cdot}51$

and those for 1948 June 17·1 and 1950 August 2·4 were finally selected for use. (No. 94 was from a plate described as "poor", and was rejected for this reason.) Some thought was given to the possibility of tracing a mean between No. 43 and No. J65081; but the Johannesburg plates (J-numbers), in contrast to the Washington ones, have no independent zero of position-angles on them, and in addition they were often rather underexposed for Watts' machine. (They had not originally been taken for this purpose.) The idea was therefore abandoned. The maximum distance of the limb, on trace No. 43, from the limb at the eclipse occurs in lunar P.A. 235° (reached by the "leader" cusp at the start of the run), and is about $0^\circ{\cdot}56$ of lunar surface distance; the maximum distance for trace 275 is in lunar P.A. 156° (reached by the "follower" cusp at the end of the run), and is about $0^\circ{\cdot}34$. The large-scale fit with the eclipse results is undeniably poorer towards the end of the run, but it is by no means certain that the major part of this misfit is even due to this cause at all; on the whole, it seems to be true that misfits of this magnitude cause little real difficulty, though no doubt they would affect the detail shown in Fig. 1.

The method of using the traces is as follows. For any given time, reckoned from mid-eclipse, the half-length, ψ, of the crescent (expressed as an angle at the Moon's centre, not the Sun's) is calculated, and also the value of $M - M_0$. Since M_0, and also C, the position-angle of the Moon's axis (as viewed at Mombasa), are both known, the lunar position-angles, Λ and Φ, of the leader and follower cusps for a smooth Moon, can be computed for this time from

$$\Lambda, \Phi = M_0 + 180^\circ - C + (M - M_0) \pm \psi, \tag{15}$$

where the upper sign refers to the leader, since position-angles were increasing. In order to establish the effect of the limb features at the leader cusp (say) on the P.A. for this time, one turns to the appropriate trace (west limb), and lays down, through the point Λ on the baseline, a line which slopes upwards at the correct angle to represent the Sun's limb. One then reads off the difference in lunar P.A., $E'(\Lambda)$ say, between Λ and the P.A. where this line cuts the trace. One does the same for the follower cusp, at the Φ corresponding to the same $t - t_0$ value; and the quantity $E' = \frac{1}{2}[E'(\Lambda) + E'(\Phi)]$ is (except for a small correction) the net effect which the lunar limb irregularities have caused. It should be directly comparable with the value of $P - M$ for this same $t - t_0$ value.

The procedure may be clearer if a couple of practical examples are taken. Figs. 5 and 6 show two stretches of Watts' traces; the continuous lines represent the trace and its baseline, with true lunar position-angles as marked off by Watts, and the short straight lines are lines ruled on a protractor of transparent plastic, of which a number were specially prepared for this work. The horizontal protractor lines are merely guides for orientation. The vertical lines are ruled at separations of $\frac{1}{10}$ degree on the mean scale of Watts' abscissae; his actual

scale was very slightly variable, on account of differential refraction and other minor factors, but was sufficiently uniform to make it easy to adjust by eye, within any one degree, for the slight error involved in using a constant protractor spacing. The slanting line represents the Sun's limb; the slant depends on the sharpness of the cusp (reckoned for a smooth Moon), and this varies during the eclipse and is also changed systematically if the minimum apparent separation of

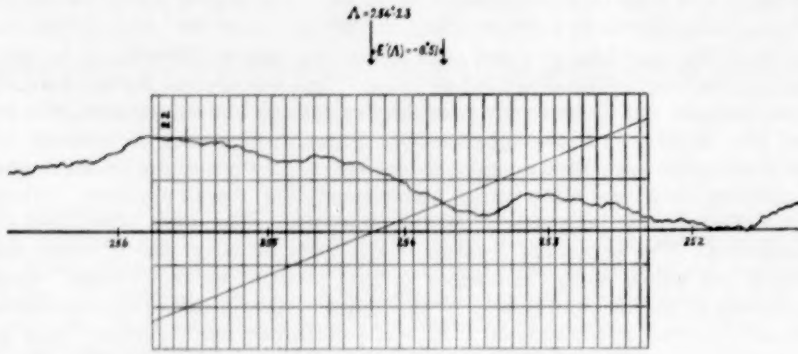

FIG. 5.

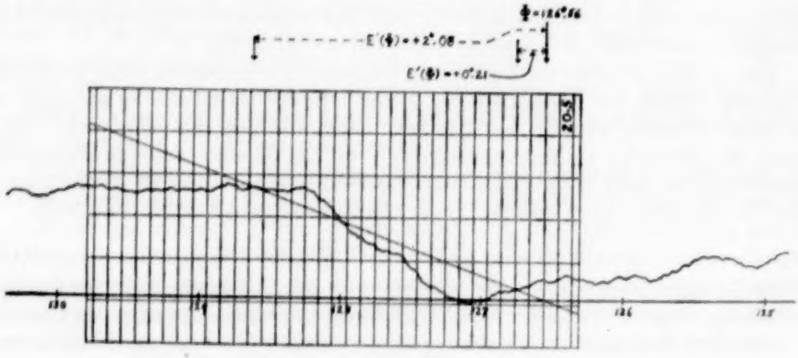

FIG. 6.

centres is changed; in addition it depends on the ratio of vertical and horizontal scales for the trace in question, which depends on the Moon's distance at the time the plate was taken. The computed slants (both traces included) ran from $20^\circ\!\cdot 5$ to $27^\circ\!\cdot 5$, and experience showed that the nearest half degree was sufficiently accurate; 15 protractors in all were therefore prepared, and a critical table was computed for each of the two traces, to indicate the ranges within which each protractor should be used.

Fig. 5 shows the leader cusp at a point where it is working its way up a fairly steep slope. The value of Λ, which would in practice be computed for the $t - t_0$ corresponding to one of the 310 normal times of the chronograph record, is here taken as $254^\circ\!\cdot 23$ and the appropriate protractor is laid down so that (*a*) a horizontal line is conveniently close to, and strictly parallel to, the baseline;

and (*b*) the intersection of the slant line and a vertical line coincides with the point A on the baseline. The difference between A and the P.A. where the slant line cuts the trace (P.A. of the true cusp) is then read off directly, in tenths and hundredths of a degree of protractor-ruling, as $-0^\circ{\cdot}51$; this is $E'(A)$ for the time in question. It is clear that the P.A. of the true cusp will be increasing, in this region, more slowly than that of the "smooth" one, and that the rate will go up to that for the smooth Moon again as soon as the profile becomes level, whether it is then high or low.

In Fig. 6 the follower cusp is shown at a point where a Baily's bead has separated off from the cusp but has not yet disappeared. The computed P.A. for the "smooth" cusp is taken as $\Phi = 126^\circ{\cdot}56$, and the distance to the actual one is either $+2^\circ{\cdot}08$, if the "main" cusp is read, or $+0^\circ{\cdot}21$ if the bead is. (The inner edge of the bead, at $E'(\Phi) = +1^\circ{\cdot}45$ in this case, was very seldom a good object on the film.) It is evident that in determining E'-values, just as in measuring the film, as the computed Φ increases one will at some point or other be forced to jump discontinuously forward, from what was once the main cusp but has now become a negligible bead, and to set on a new "main cusp" which had previously been only a nick in the crescent. The decision when to jump was guided (on both the leader and follower traces) by what had been done in measuring the film originally; but a further special study was afterwards undertaken (see below) which disposes of the risk that one might in this way force the E curve into agreement with the $P-M$ one to an extent which the data themselves did not warrant. (Of the 310 normal P-values, only about 26 were subject to any uncertainty on this account, so that the maximum error which could be introduced is not in any case large; but this revision practically eliminates it). It is clear that steep down-slopes on this trace have the same effect as steep up-slopes on the other, i.e. they will introduce a downwards slant in the E-curve.

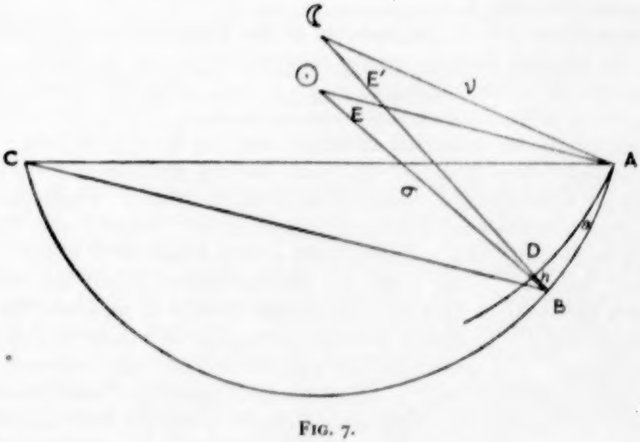

FIG. 7.

The true effect of limb features is slightly different from E'. This may be seen as follows. In Fig. 7, ⊙ and ☾ are the centres of the Sun and Moon, σ and ν are their radii, θ is the cusp-angle (computed for a smooth Moon), and h is the height of a lunar mountain above the smooth level. A is where the (leader)

cusp would be if the Moon were smooth, B is the actual cusp, and C is the other cusp, which need not lie on the smooth Moon's limb but necessarily does lie on the Sun's. If the Moon's limb were smooth at the leader cusp, AC would be the line of cusps, while the actual line is BC; the true effect produced in the P.A. of the line of cusps, by the mountain of height h, is thus $A\widehat{C}B$ or $\frac{1}{2}A\widehat{\odot}B = \frac{1}{2}E(\Lambda)$ say. The angle read off from the trace, $E'(\Lambda)$, is $A\widehat{☾}B$, and if BD is drawn perpendicular to the smooth limb, we have

$$AD/\nu = (h/\nu)\cot\theta = E'(\Lambda)$$

and

$$AB/\sigma = (h/\sigma)\operatorname{cosec}\theta = E(\Lambda),$$

so that

$$E/E' = (\nu/\sigma)\,.\,\sec\theta; \tag{16}$$

the effect introduced by the mountain at the leader cusp is thus

$$\tfrac{1}{2}E(\Lambda) = \tfrac{1}{2}E'(\Lambda)\,.\,(\nu/\sigma)\,.\,\sec\theta.$$

Exactly the same argument applies, independently, to the follower cusp, and the total effect on the line of cusps is thus

$$E = \tfrac{1}{2}[E'(\Lambda) + E'(\Phi)](\nu/\sigma)\,.\,\sec\theta. \tag{17}$$

For this eclipse, ν/σ was 1·015, constant to the accuracy needed here; θ varied from $4^\circ{\cdot}4$ down to $3^\circ{\cdot}2$ and up to $4^\circ{\cdot}2$ again, and $\sec\theta$ may thus also be considered constant, at 1·002. We have then

$$E = 0{\cdot}5085[E'(\Lambda) + E'(\Phi)]. \tag{18}$$

Although the difference between 0·5085 and the simpler $\frac{1}{2}$ is small, it can enter systematically into the residuals $P-M-E$, since neither the mean height nor the mean slope of the E curve is necessarily zero; the rigorous factor has therefore been used in the reductions. It is readily verified that we may ignore the fact that the slant lines on the protractors should ideally be slightly curved (since $\nu \neq \sigma$); even when $E = 1^\circ$, the error introduced into E by this is less than $0^\circ{\cdot}001$, and it is quasi-random.

The question which of the features at the leader cusp will combine with which at the follower depends on the crescent length, 2ψ, and in fact the form of the function E is fairly sensitive to ψ; ψ itself is decidedly sensitive to the difference, $\nu - \sigma$, of the lunar and solar semi-diameters. In the first trial solutions which were made, for restricted stretches only, by R. d'E. A., the "eclipse" value of the lunar semi-diameter was used, and the resulting E curve displayed discouragingly little detailed resemblance to the $P-M$ one. Inspection showed that a shorter crescent would probably combine the "leader" and "follower" features more suitably, and a re-trial with a semi-diameter 1″ larger confirmed this. The "eclipse" value, given by $\sin\nu = 0{\cdot}272274\sin\pi_{☾}$, was derived (**4**) from eclipse observations, in a quite deliberate attempt to represent the effect of the "average valley", at second and third contacts, by means of a circular but undersize Moon. The preoccupation with second and third contacts (**5**) even led to its re-introduction after it had been abandoned, and that preoccupation still persists, of course; but a false semi-diameter is not the most suitable way to cater for it if detailed contours are available, since these must be applied to the true mean. (A false value smaller than the true one is in any case unsuitable for first and fourth contacts, and even for second and third contacts in annular eclipses, since these are all concerned with the "average mountain" and not with the "average valley".)

As soon as a larger semi-diameter than the "eclipse" one was seen to be demanded by the present work, it was decided to use the "occultation" one, given by $\sin \nu = 0{\cdot}2724953 \sin \pi_{☾}$. (The one used in the lunar ephemeris, given by $\sin \nu = 0{\cdot}272481 \sin \pi_{☾}$, or $\nu = 0{\cdot}272446 \pi_{☾} + 0''.079$, would be equally satisfactory.) Fig. 8 indicates, for one particular feature, how much the agreement between E and $P-M$ is improved by the change; in general (except where one limb was nearly featureless), a similar improvement resulted throughout.

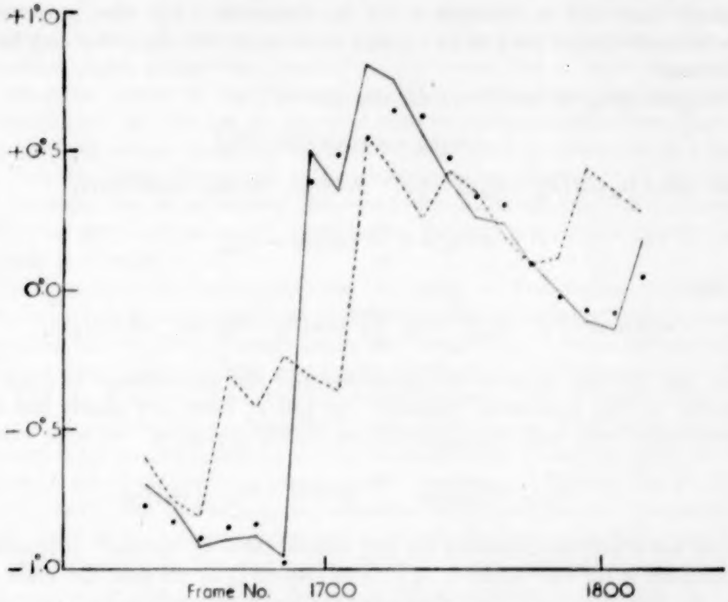

FIG. 8.—*Comparison between "Eclipse" and "Occultation" semi-diameters.*
——— *E curve (occultation semi-diameter).*
- - - - *E curve (eclipse semi-diameter).*
• • • *P−M values (Mombasa).*

The values of ψ and θ can be computed as in (**1**), but it is more satisfactory to work exclusively with the data given in the "eclipse" pages of the Almanac, using the formulae given in the N.A. "Explanation" in 1940 and previous years. The necessary amendment, to take account of the larger value of ν, is to increase l_1, and decrease l_2, by $(0{\cdot}2724953 - 0{\cdot}2722740) \sec f/(1-b)$, where $b = \sin \pi_{\odot}/\sin \pi_{☾}$, whether l_2 is positive or negative. (It is negative in all eclipses which are total throughout.) $\operatorname{Tan} f$ is not affected, to the accuracy required. The eclipse pages then contain all the necessary data, and they obviate the need for an appreciable amount of interpolation, as well as being free from irradiation terms, and already incorporating the special eclipse corrections (**6**).

Accordingly we have

$$\cos \psi = \frac{m^2 - L_1 L_2}{m(L_1 - L_2)}, \tag{19}$$

where $L_1 = l_1 - \zeta \tan f_1$, $L_2 = l_2 - \zeta \tan f_2$.

A large number (310, in the present case) of ψ-values are needed for insertion in (15); however, it is not necessary to compute a large number of them from this formula; it is appreciably simpler to compute only enough to determine the constants in the formula

$$\psi = a_0 + a_1\cos(M-M_0) + a_2\cos 2(M-M_0) + b_1\sin(M-M_0) + b_2\sin 2(M-M_0) \quad (20)$$

which fits the rigorous values with ample accuracy, and then to compute from this an ephemeris of ψ-values for the whole degrees of $M-M_0$ (or for every alternate one) and to interpolate for the fractions. For this purpose it is convenient to obtain $\cos\psi$ as an explicit function of $M-M_0$; this may be done as follows.

Neglecting s_0, we have from the real part of (12)

$$m = (m_0 + dm)\sec(M-M_0), \quad (21)$$

where $dm = \frac{1}{2}r_0\cos(R_0 - M_0)(t-t_0)^2$. Writing, in this small term,

$$t - t_0 = +\frac{m_0}{n_0}\tan(M-M_0),$$

we have

$$m \simeq m_0\sec(M-M_0)\left[1 + \frac{1}{2}\frac{r_0 m_0}{n_0^2}\cos(R_0 - M_0)\tan^2(M-M_0)\right], \quad (22)$$

which can be used in both the numerator and the denominator of (19). The quantity $L_1 - L_2$ is sensibly constant. L_1 and L_2 both vary slowly and almost linearly with time, and $d(L_1L_2)/dt$ can be readily evaluated; we have then

$$L_1L_2 = (L_1L_2)_0 + \frac{m_0}{n_0}\tan(M-M_0)\frac{d}{dt}(L_1L_2), \quad (23)$$

so that $\cos\psi$ can be computed for any desired $M-M_0$ values. It is sufficient to compute it for the values 0, $\pm v$, $\pm 2v$, where $\pm 2v$ lie near the ends of the run; the five values of ψ then determine the five coefficients in (20) very simply.

The angles of slant of the protractors are computed as follows. If one imagines a lunar feature which slopes upwards, with respect to the smooth limb, at an angle θ, this slope will of course be unaltered by changing the Moon's distance. But since Watts' abscissae are position-angles, i.e. are not affected by distance, while his ordinates are seconds on the sky, the mountain will appear both higher and steeper, *on his traces*, when the Moon is near. His horizontal scale, being 1° of P.A. to 4·06 cm, is 1 cm $= \nu/232{\cdot}7$ when the semi-diameter is ν (in seconds of arc). His vertical scale being 1 cm $= 0''{\cdot}605$ (for the focal length of the Washington telescope), a mountain whose true slope is θ will have a slope, γ, on his trace given by

$$\tan\theta = (140{\cdot}8/\nu)\tan\gamma, \quad (24)$$

where ν is the topocentric semi-diameter when the Washington plate was taken. If now we interpret θ as the cusp angle at the eclipse, we can interpret γ as the appropriate protractor slant for reading the trace, since the tangents of all slopes are affected in the same proportion.

Since

$$\sin\theta = \frac{2m}{L_1 + L_2}\sin\psi, \quad (25)$$

we may compute an empirical formula for $\sin\theta$ in terms of $M-M_0$, by methods analogous to those of the last section. In practice, it appeared that a symmetrical formula was sufficient, i.e.

$$\begin{aligned}\sin\theta &= a'_0 + a'_1\cos(M-M_0) + a'_2\cos 2(M-M_0)\\ &= a'_0 - a'_2 + a'_1\cos(M-M_0) + 2a'_2\cos^2(M-M_0),\end{aligned} \tag{26}$$

so that $\cos(M-M_0)$ can be expressed as a function of $\tan\gamma$, thus enabling critical tables to be explicitly computed, for selected values of the protractor angle γ.

Historical.—In the present case, all of this analysis, and the accompanying measurement of the limb traces, was carried through several times. The zero of position-angles which was originally used demanded a large change from the 'Almanac' place of the Moon; Dr Watts made important revisions of his 'baselines'; the change to the final zero of position-angles cancelled most of the original change in the Moon's place; and each of these changes was so large that the limb-effects had to be completely re-assessed. In any future work, however, one measurement should be fully sufficient, and the total amount of labour is then not excessive, considering the accuracy which appears to be attainable as a result.

The first analyses were made on the basis of limb-traces supplied, by Dr Watts, before his own work had reached the stage of collating different traces. It was then evident that although (once the "occultation" semi-diameter of the Moon had been adopted) there was a great similarity in the features of the $P-M$ curve and the E curve, there was a systematic term in the ordinate differences, the central regions being a considerable fraction of 1° apart if the ends coincided; there was also a general tendency for the features of the E curve to lie to the right of those of the $P-M$ curve. Owing to the "sawtooth" form of the $P-M$ and E curves, and to the fact that at this stage the steep up-slopes had practically no weight, since on both curves there was some arbitrariness in deciding just when they should occur (p. 65 and p. 79 above), it was difficult to estimate the phase shift, even when the general run of the separation in ordinates had been approximately assessed from the various horizontal stretches; but it was clear that the phase displacement lay roughly between $0^\circ{\cdot}1$ and $0^\circ{\cdot}5$, i.e. that there was a discrepancy of this magnitude between the zero of position-angles used in this work and that used by Dr Watts, in addition to some unknown systematic factor which falsified the ordinates.

When Dr Watts saw these results, at the time of the Rome I.A.U. meetings, he immediately realized that the systematic difference in ordinates might be connected with the difference between the original baselines of his traces and the "revised" ones which he was then engaged in obtaining. His original "baseline" (zero from which mountains reckon upwards and valleys down) is a nearly straight line, computed and drawn by him on the recording paper, and representing in effect the best arc of a circle that can be fitted to the particular limb photograph in question, after correcting for the distortion produced by differential refraction; it is obtained by a straightforward least-squares process based on that trace alone, without reference to any others. Very near full Moon, a large fraction of the whole perimeter may be illuminated (though even then there is almost always some "defective illumination", and when the libration in latitude is large there is always a good deal); at most times of the month, however, the usable arc is much shorter, often only 160° or so, and as there are

very extensive regions which lie entirely above, or below, the mean height, the arc of a circle which best fits any particular photograph may differ considerably from the complete circle which would be fitted to the complete perimeter, supposing that this could be, and were, all illuminated at the same time. The regions beyond the poles never can be illuminated when they are on the limb, and thus although one can try to construct complete circles, by fitting together overlapping arcs taken at different phases but with the same l and b values, these circles cannot be closed, however many photographs one may take. The best complete circle for any (l, b) pair may also differ from any central section of that sphere which best fits the Moon as a whole, and it is clearly desirable to refer all profiles to this sphere if possible. This involves some intricate considerations, and a great deal of laborious intercomparison between profiles photographed when the libration coordinates (l, b) were different. Dr Watts is still engaged on this difficult task; he very kindly advanced the particular traces used in the present work ahead of his regular schedule, and communicated "revised" baselines for them specially, the general framework of his intercomparisons being sufficiently far advanced to make this possible, though a further revision (to take account of the difficult problem of "dead ground") is still in progress. The "revised" baselines depart from the original ones over fairly long stretches, sometimes falling as much as $0''\cdot35$ above or below them; it is important to realize that the employment of detailed limb-traces, even of the high quality which the Washington ones possess, will therefore not necessarily eliminate discrepancies of this magnitude between the Moon's place at different librations, at least if the "place" is based on observations of only a short length of limb; it is only the "revised" baselines which will make it possible to eliminate them, and even then not completely unless the centre of the sphere to which they are all referred coincides with the centre of gravity.

A re-computation using the new baselines gave a somewhat different place for the Moon (the mean values of the changes not being zero), and also a considerably better overall fit; for what it is worth, this improvement undoubtedly confirms the general correctness of Watts' revision, but it is clear from Fig. 9 that some systematic discrepancies remain. The longer the arcs of the Moon's limb that are observed in eclipses, the more chance there is, of course, for any residual curvature error in the baselines to show up; if only a short stretch of limb is used, such errors cannot be detected, and the apparent probable error may be correspondingly less than the real uncertainty. However, the small curvature difference still remaining in Fig. 9 may also have its origin in the present work.

The considerable improvement which had now been obtained only hardened the discrepancy in abscissae already noted, and there was extensive correspondence with Dr Watts on the subject. He had not originally expected that his position-angles would be so reliable, and so valuable, as they have proved to be, and his original method of determining them had been somewhat less rigorous, and less complete, in its analysis than his data actually turned out to warrant. In the course of the correspondence, he discovered and communicated several small systematic corrections of P.A. which, if they had all had their maximum values, and in the right sense, in the two traces in question, could almost have accounted for the whole discrepancy; however, the net effect (which varies from trace to trace) is less, and on both traces it proved to be definitely insufficient,

though it did produce some improvement. Ultimately, both Dr Watts and the senior author were satisfied that no further corrections could be expected; the remaining error must then be looked for in the zero of position-angles used in the present work. If this zero is increased, all the values of P are increased also, and it becomes necessary to increase the Moon's longitude in order to lower the $P-M$ curve to the old level again; the values of M for all the times t are thus increased, and the values of Λ and Φ are increased with them; any given feature on the lunar limb is therefore reached at an earlier time, so that the E curve is displaced to the left.

The possibility that this might be the correct thing to do was confirmed by the fact that the analyses had so far all resulted in a lunar longitude smaller by several tenths of a second of arc than the value obtained from occultation results; it became increasingly clear, as the occultation reductions themselves proceeded, and were supported by meridian work, that this discrepancy was probably real, and unacceptable. Both the discrepancy with Watts' position-angles and the discrepancy in the Moon's place thus suggested that the zero used so far was too small, i.e. the values of P associated with any given absolute time ought to be larger. Serious consideration was given to the possibility of discarding the zero based on the supposed diurnal run altogether, and basing the analysis on that of Watts' traces; but at that time his estimate of the probable error of his position-angles was about $\pm 0^\circ\cdot 1$, which would mean $\pm 0''\cdot 1$ in the corresponding coordinate of the Moon's place, and this is much larger than the present method appears capable, in principle, of yielding.

It was at this stage that a detailed reassessment showed that the slope of the second part of Fig. 3 indicated a zero which would very nearly cancel both the above stubborn discrepancies at the same time. Since the scatter of individual points was also less in the final stretch than in the earlier part, it was decided to adopt this latter run as the true direction of diurnal motion, ignoring the first part altogether. The solution (2) obtained in this way is formally rigorous, with a reasonably small probable error, and is independent of Watts' zero (though in harmony with it); it contains an element of arbitrariness in its basic assumption, but any other treatment of the points in Fig. 3 (including discarding them altogether) would involve some arbitrariness, and all would result in much larger formal uncertainties.

Before the final analysis was undertaken, it was decided to eliminate as far as possible the arbitrary factor (pp. 65, 79) in the moments of transfer from "cusp" to "appendage", or the reverse, both for the upward jumps in P and for those in E. Accordingly, at all points where either a bead or a conspicuous thread-like appendage appears at the leader-cusp (or disappears, at the follower) the film was now examined frame by frame, to see which was the extreme frame on which it could be detected. The $(t-t_0)$ for this particular frame was taken, and the Λ (or Φ) value for this was computed just as it had been for the 310 means of ten; this Λ (or Φ) was then compared with the extreme Λ (or Φ) at which the appropriate protractor had to be laid down on the limb-trace, so as to bring the bead or thread to the verge of appearance (or disappearance) there also. The differences between the values indicated by the photographs and those indicated by the Washington trace were meaned for the Λ trace, and similarly for the Φ trace, and these two means were adopted as corrections to the position-angle systems of those traces.

It should perhaps be stressed that whatever these corrections may be, they do not directly affect the final result. Their mean indicates a systematic rotation of the Moon's outline (as compared with Watts), but *not* a rotation of the crescent and therefore not a change in the Moon's place; their difference indicates either a small difference between the P.A. systems of the two traces, as determined by Watts, or else a correction to the computed ψ values, which would mean a correction to $\nu-\sigma$; but the different possibilities cannot well be separated. For any trace obtained (as No. 275 was) near the opposite node to the eclipse one, the P.A. correction now derived is also not effectively separable from a correction to the inclination of the Moon's axis. The present method is therefore not suitable for determining these corrections; but this was never its purpose.

The adoption of these two corrections means that the details of the shape of the E curve are now practically "frozen"; the phase relationship between the leader trace and certain critical frame-numbers has been discovered, and similarly for the follower, so that their (mean) relationship to each other can no longer be varied. Even though the least-squares solution may lead to an appreciable value of the correction Δm_0, and so to a change in the minimum apparent separation of centres and in the computed crescent length 2ψ, the only effect will be corresponding changes (of opposite signs) in the corrections that must then be inferred for the P.A. systems of Watts' two traces; similarly, a change in the mid-eclipse time will necessitate changes of the same sign in both corrections. The E curve will be essentially unchanged and undisplaced, and in particular it need not be recomputed, after the least-squares solution, in order to evaluate the residuals (at least if the corrections to M_0 and m_0 are as small as they are here); the E curve of Fig. 9 is that on which the least-squares solution was based, and also that from which the final residuals were obtained, for computing probable errors. These considerations thus result in an appreciable saving of labour, at the same time that they also remove the uncertainty in the phase relation between the steep up-slopes of the E curve and those of the $P-M$ curve; the difference between the ordinates of the two curves has now just as much weight at these places as it has elsewhere. The method is, of course, only practicable if the definition in the photographs is sufficient to show detail at the cusps.

The actual values for the corrections to Watts' systems, after the M_0 and ψ values used in (15) had been corrected by the results of the least-squares solution, became $-0^\circ{\cdot}03 \pm 0^\circ{\cdot}03$ for trace 43 and $+0^\circ{\cdot}09 \pm 0^\circ{\cdot}02$ for trace 275; the mean of these would clearly be acceptable even if the probable errors of both of Watts' systems and of the Mombasa one were all zero. Both the corrections and their probable errors were slightly smaller, after the least-squares solution, than they had been at the "first approximation". They would, of course, both have to be about $-0^\circ{\cdot}3$ if the original zero of position-angles were correct.

The above argument assumes that there is no systematic error due to irradiation, as between a "thread" or other faint cusp feature, on the one hand, and a blunt cusp caused by steep limb-slopes "towards" the crescent, on the other. Blunt tips correspond at both cusps (p. 65) to down-slopes in the $P-M$ curve, and if it were the case that the photographic image of a blunt tip was always somewhat extended, by irradiation, as compared with the image of a thread (which, in the limit, is underexposed), the values of P would be abnormally great when the leader cusp was blunt, i.e. on those down-slopes for which the leader cusp was responsible, and abnormally small for the follower

cases. There clearly are some down-slopes in Fig. 9 where $P-M>E$, and others where $P-M<E$, systematically; but examination showed no clear correlation between the sign of $P-M-E$ and the question whether it is the $E(\Lambda)$ values, or the $E(\Phi)$ ones, which show a strong downwards drift in these regions. It thus seems satisfactory to adjust the phase relations of $E(\Lambda)$ and $E(\Phi)$ on the basis of the sudden up-slopes only; observationally, they are the only directly identifiable features.

The equation of condition.—Neglecting higher-order terms, eqn. (10) is

$$m\,e^{iM}=m_1\,e^{iM_1}+n_1\,e^{iN_1}(t-t_1), \tag{27}$$

and we require the effect on M, at any given t, of making small changes in the parameters m_1, M_1, etc., which are the values taken by m, M, etc. at the fixed arbitrary time t_1. Accordingly, if Δ denotes the effect, at any given frame-number (or time) of all such changes as we may make in the positions of the Moon, observer, etc., we have

$$\left(\frac{\Delta m}{m}+i\Delta M\right)m\,e^{iM}=\left(\frac{\Delta m_1}{m_1}+i\Delta M_1\right)m_1\,e^{iM_1}+\left(\frac{\Delta n_1}{n_1}+i\Delta N_1\right)n_1\,e^{iN_1}(t-t_1). \tag{28}$$

Having differentiated, we may change all the subscripts to zeros, i.e. we may write the equation for mid-eclipse. It should be noted, however, that Δm_0, ΔM_0, etc. are not then (in general) the changes in the mid-eclipse values; they are the changes in the values at the fixed time t_0. In particular, the change in the mid-eclipse value of M is almost identically zero, since mid-eclipse is defined by $M=N\pm\pi/2$ and both n and N are very insensitive to any change in the basic data, or in the time. Indeed, this insensitivity is such that we may at once drop Δn and ΔN in (28), and write

$$\Delta m+im\Delta M=\left(\frac{\Delta m_0}{m_0}+i\Delta M_0\right)m_0\,e^{-i(M-M_0)}. \tag{29}$$

In order to eliminate m from this, it is convenient to define the angle T by the equation (essentially eqn. (4))

$$\tan(T-M_0)=\frac{n_0}{m_0}(t-t_0), \tag{30}$$

which differs from (13) only by terms in r_0. T is thus a good approximation for M; the difference amounts to only a very few minutes at the ends of the 76° run. Dropping the terms in r_0 and s_0 from (12), and replacing M by T, the real part becomes

$$m\simeq m_0\sec(T-M_0). \tag{31}$$

Inserting this in (29) and replacing M by T there also,

$$\frac{\Delta m}{m_0}+i\sec(T-M_0)\Delta M=\left(\frac{\Delta m_0}{m_0}+i\Delta M_0\right)e^{-i(T-M_0)}. \tag{32}$$

The imaginary part of this is

$$\Delta M=\cos^2(T-M_0)\Delta M_0-\cos(T-M_0)\sin(T-M_0)\frac{\Delta m_0}{m_0}, \tag{33}$$

and this yields the necessary equation of condition, since we may equate ΔM to the residuals, $P-M-E$, obtained by comparing the observations with the Washington traces. The equation is therefore

$$\cos^2(T-M_0)\Delta M_0-\cos(T-M_0)\sin(T-M_0)\Delta m_0/m_0=P-M-E, \tag{34}$$

where M is computed from (13), i.e. including terms in n_0 and r_0, and T is defined by (30).

The normal equations which result from this are

$$[c^4]\Delta M_0 - [c^3s]\Delta m_0/m_0 = [c^2(P-M-E)]$$
$$+[c^2s^2]\Delta m_0/m_0 = -[cs(P-M-E)], \qquad (35)$$

where we have written c for $\cos(T-M_0)$ and s for $\sin(T-M_0)$; the two "unknowns" ΔM_0 and $\Delta m_0/m_0$, with their weights, may then readily be evaluated.

In order to use the result, we return to (8). Evidently

$$\Delta(x-\xi) = m\cos M\Delta M + \sin M\Delta m$$
$$= (y-\eta)\Delta M + (x-\xi)\Delta m/m,$$

and in particular

$$\Delta(x_0-\xi_0) = (y_0-\eta_0)\Delta M_0 + (x_0-\xi_0)\Delta m_0/m_0. \qquad (36)$$

Similarly $$\Delta(y_0-\eta_0) = -(x_0-\xi_0)\Delta M_0 + (y_0-\eta_0)\Delta m_0/m_0,$$

and we can thus correct either x_0 or ξ_0, and either y_0 or η_0, from the results of (35).

The real part of (32) also gives an equation of condition, and this would be applicable to observations which gave m (essentially the apparent separation of centres of Sun and Moon) as a function of time, instead of to observations of M. However, the two "unknowns" are the same as before, namely ΔM_0 and $\Delta m_0/m_0$; thus, since all information which eclipses can give about the relative places of Sun, Moon, and observer must be expressible in terms of corrections to the run of m and M, it appears that eclipse observations must in effect always yield only the combined corrections $\Delta(x_0-\xi_0)$ and $\Delta(y_0-\eta_0)$, without separating Δx_0 from $\Delta\xi_0$ or Δy_0 from $\Delta\eta_0$, at least unless higher-order terms of sufficient weight can be introduced, so as to get further unknowns in the equation of condition. This seems unlikely, so far as methods confined to a total time of a few minutes near mid-eclipse are concerned. Equation (32) is not dependent at all on the present method of observation; it is equally applicable to any method whatever, inside the shadow track as well as outside, subject only to the limitations within which the approximation $M \simeq T$ holds, i.e. to the condition that the time shall be close to mid-eclipse. (The approximations $\Delta N_0 = \Delta n_0 = 0$ will always hold if the corrections to the places of Moon, Sun, or observer are small; they represent second-order terms.) The question whether to set Δx_0 or $\Delta\xi_0$ to zero (or to some other figure) in order to derive a value for $\Delta\xi_0$ or Δx_0, as the case may be, will thus usually have to be made on other grounds.

Numerical values.—For the final analysis one small further change was made. The Besselian elements used hitherto had been those of the A.E., very slightly smoothed so as to fit them to appropriate polynomials. However, Mrs Gossner, of the A.E. staff, reported that there was an approximation still in use when the 1948 Ephemeris was computed which had a systematic effect in the sixth decimal place. It is known, of course, that the sixth place may be several units in error on account of the limitations of Brown's Tables themselves; but it seemed preferable to remove, even so, an error which represented a systematic departure from those Tables. Mrs Gossner very kindly re-computed the Besselian elements for 03^h, 04^h, and 05^h U.T., correcting this error; similar computations were then carried out for 06^h, and for $04^h\,30^m$, by C.A.M. (The elements for $04^h\,30^m$ were computed from α and δ values interpolated to this time from the hourly values.) The values of α and δ used were those furnished by the N.A.O. for eclipse purposes, which go formally to $0''\!\cdot\!01$ in both

coordinates, and for all five times the special eclipse corrections to the places of Sun and Moon, as adopted in the Almanacs, were also included, but carried to $0''\cdot01$ throughout instead of rounding off to $0''\cdot1$ at an early stage. (They were still taken as constants in α and δ though in principle it is $L_{\odot}$, $L_{☾}$, and $B_{☾}$ which should be taken as constant.)

The x and y values so obtained were as follows:—

U.T. =	$03^h\,00^m$	$04^h\,00^m$	$04^h\,30^m$	$05^h\,00^m$	$06^h\,00^m$
$x =$	$-1\cdot6765355$	$-1\cdot1628069$	$-0\cdot9058756_2$	$-0\cdot6489093$	$-0\cdot1349067$
$y =$	$+0\cdot3744145$	$+0\cdot1410343$	$+0\cdot0244052_7$	$-0\cdot0921730$	$-0\cdot3251808$

(37)

They were fitted to polynomials in at least seven figures, to facilitate checking and to minimize rounding-off errors; the polynomials were obtained using $04^h\,30^m$ as origin, with a fourth-power term included, and after transformation to the origin $t_1 = 04^h\,28^m$, close to the time ($04^h\,27^m\cdot9$) of mid-eclipse, they were

$$\left.\begin{aligned} x &= -\cdot9230056 + \cdot00856493(t-t_1) + 1\cdot98\times10^{-8}(t-t_1)^2 - 5\times10^{-11}(t-t_1)^3, \\ y &= +\cdot0321790 - \cdot00388692(t-t_1) + 2\cdot83\times10^{-8}(t-t_1)^2 + 2\times10^{-11}(t-t_1)^3, \end{aligned}\right\} \quad (38)$$

where t is in minutes.

The following values were also obtained directly, for $t = t_1$:—

$$\mu = 251^\circ\cdot0884\,; \quad \dot{\mu} = 4\cdot36378\times10^{-3}\ \text{(radians/min)}; \qquad (39)$$

$$\sin d = -\cdot2485986, \quad \cos d = +\cdot9686066, \quad \dot{d} = -3\cdot73\times10^{-6}. \qquad (40)$$

For l_1, l_2, f_1, and f_2, the A.E. values were used without change except for that required by the adoption of the "occultation" semi-diameter of the Moon in place of the "eclipse" one. The values were (6)

$$\left.\begin{aligned} l_1 &= +\cdot5441576 + 2\cdot355\times10^{-6}(t-t_1); \quad \tan f_1 = +\cdot0047120; \\ l_2 &= -\cdot0021925 + 2\cdot355\times10^{-6}(t-t_1); \quad \tan f_2 = +\cdot0046885. \end{aligned}\right\} \quad (41)$$

The adopted geographical coordinates of the station were

$$\lambda = -39^\circ\cdot6800\,; \quad \phi = -4^\circ\cdot0702\,; \quad h = 14\cdot7\ \text{metres};$$

and using the ellipticity ($\frac{1}{297}$) adopted in the Almanac these lead to

$$\rho\cos\phi' = +\cdot9974972, \quad \rho\sin\phi' = -\cdot0705028 \qquad (42)$$

and to

$$\xi = -\cdot9326812 + \cdot00154349(t-t_1) + 8\cdot8804\times10^{-6}(t-t_1)^2 - 4\cdot90\times10^{-9}(t-t_1)^3, \qquad (43)$$

$$\eta = +\cdot0196408 + \cdot00101314(t-t_1) - 0\cdot8226\times10^{-6}(t-t_1)^2 - 3\cdot22\times10^{-9}(t-t_1)^3, \qquad (44)$$

$$\zeta = +\cdot3601268 + \cdot00394217(t-t_1) - 3\cdot2658\times10^{-6}(t-t_1)^2 - 12\cdot51\times10^{-9}(t-t_1)^3. \qquad (45)$$

Thus

$$\begin{aligned} x-\xi = {}& +\cdot0096756 + \cdot00702144(t-t_1) - 8\cdot8606\times10^{-6}(t-t_1)^2 \\ & + 4\cdot85\times10^{-9}(t-t_1)^3, \end{aligned} \qquad (46)$$

$$\begin{aligned} y-\eta = {}& +\cdot0125382 - \cdot00490006(t-t_1) + 0\cdot8509\times10^{-6}(t-t_1)^2 \\ & + 3\cdot24\times10^{-9}(t-t_1)^3, \end{aligned} \qquad (47)$$

from which we may deduce, in the usual way, the "Almanac" time of mid-eclipse, $t_a = 04^h\ 27^m{\cdot}911\ 20$ and the "Almanac" mid-eclipse values

$$(x_a - \xi_a) = +{\cdot}009\,052\,0, \quad (y_a - \eta_a) = +{\cdot}012\,973\,4.$$

The difference between these values and the corresponding ones based on the original A.E. elements is of course slight.

The corrections (to the original A.E. elements) demanded by all the earlier analyses were rather large. When the zero of position-angles was finally changed, a new "first" approximation was reached by adding estimated (arbitrary) corrections of the opposite sign, which were also rather large; at the same time, the elements due to Mrs Gossner were adopted. The net effect is that the final least-squares solution was based on a system given rigorously by

$$\left.\begin{aligned} x - \xi &= (46) + \Delta_1(x - \xi), \\ y - \eta &= (47) + \Delta_1(y - \eta), \end{aligned}\right\} \tag{48}$$

where

$$\left.\begin{aligned} \Delta_1(x - \xi) &= -74 \times 10^{-7}, \\ \Delta_1(y - \eta) &= -250 \times 10^{-7}, \end{aligned}\right\} \tag{49}$$

and (46) and (47) are the "Gossner" values, i.e. are derived from Newcomb's and Brown's Tables, as modified by the corrections $+1''{\cdot}00$ to the mean longitude of the Sun, $-1''{\cdot}50$ to the mean longitude of the Moon, and $-0''{\cdot}50$ to the latitude of the Moon. There is no advantage in (48), as compared with (46) and (47); either pair could serve equally well as the first approximation, and in all future cases it seems likely that the Almanac values will be perfectly satisfactory as a basis for the only least-squares solution that will be needed. In this particular case it happened that the solution was based on the intermediate system (48), but this involves no loss of rigour.

The above values lead, in the usual way, to the mid-eclipse time, for this "first approximation", $t_0 = 04^h\ 27^m{\cdot}910\ 23 = t_1 - 0^m{\cdot}089\ 77$, and transforming (48) to this origin we have

$$\begin{aligned} x - \xi &= +{\cdot}0090378 + {\cdot}00702303\,(t - t_0) - 8{\cdot}862 \times 10^{-6}(t - t_0)^2 + 5 \times 10^{-9}(t - t_0)^3, \\ y - \eta &= +{\cdot}0129531 - {\cdot}00490021(t - t_0) + 0{\cdot}850 \times 10^{-6}(t - t_0)^2 + 3 \times 10^{-9}(t - t_0)^3. \end{aligned} \tag{50}$$

The cubic terms have now no further importance, since they do not affect $x - \xi$ and $y - \eta$ for $|t - t_0| < 1^m{\cdot}6$ (the extreme values), and therefore cannot affect $\tan(M - M_0)$; they were finally dropped at this stage. Differentiating as necessary we have

$$\left.\begin{aligned} m_0 &= {\cdot}015\,794\,5; \quad n_0 = {\cdot}008\,563\,59; \quad r_0 = 1{\cdot}78 \times 10^{-5}; \\ M_0 &= 34^\circ{\cdot}905; \quad N_0 = 124^\circ{\cdot}905; \quad R_0 = 275^\circ{\cdot}5; \end{aligned}\right\} \tag{51}$$

and so from (13)

$$\tan(M - M_0) = +0{\cdot}542188(t - t_0) - 4{\cdot}91 \times 10^{-4}(t - t_0)^2 + 1{\cdot}50 \times 10^{-4}(t - t_0)^3, \tag{52}$$

from which the values of M to be used in (34), (15), etc. were computed; also, eqn. (30) evidently becomes

$$\tan(T - M_0) = +0{\cdot}542188(t - t_0). \tag{53}$$

Five values of ψ were computed, from (19), etc., for the values $M - M_0 = 0^\circ$, $\pm 18^\circ$, $\pm 36^\circ$, and were fitted by the formula

$$\begin{aligned} \psi = 83^\circ{\cdot}099 - 7^\circ{\cdot}642 \cos(M - M_0) - 1^\circ{\cdot}287 \cos 2(M - M_0) \\ - 0^\circ{\cdot}093 \sin(M - M_0) - 0^\circ{\cdot}010 \sin 2(M - M_0), \end{aligned} \tag{54}$$

from which a table of the 80 values of ψ for the whole-degree values of $M-M_0$ from -42° to $+37^\circ$ was readily computed. The values of Λ and Φ for the 310 values of $M-M_0$ were then obtained from (15), interpolating from this table to get the actual ψ-values, and using $C=+17^\circ\cdot194$ as the topocentric P.A. of the Moon's axis at mid-eclipse at Mombasa (3).

The topocentric semi-diameters of the Moon, for the two Washington traces used, were ν (leader) $=954''\cdot4$ and ν (follower) $=922''\cdot9$; accordingly from (24)

$$\left.\begin{array}{ll} 0\cdot148\tan\gamma=\tan\theta & \text{for the leader trace,} \\ 0\cdot153\tan\gamma=\tan\theta & \text{for the follower.} \end{array}\right\} \qquad (55)$$

Equation (26) took the form

$$\sin\theta=+\cdot19113-\cdot21010\cos(M-M_0)+\cdot07540\cos^2(M-M_0), \qquad (56)$$

and critical tables for γ were prepared accordingly. The two traces were then each measured at the 310 Λ (or Φ) values, with the protractors appropriate to each region, as shown by these tables, and the 310 values of E were obtained from (18).

The normal equations.—The 310 values of $P-M-E$ were combined into 31 normal values (each of which was taken as having unit weight), and these were converted to radians. Thirty-one values of c and s were computed from (53), for the appropriate means of ten values of $t-t_0$ (i.e. of 100 chronograph times). Equations (35) then became

$$\left.\begin{array}{l} 22\cdot683\Delta M_0+1\cdot043\Delta m_0/m_0=-\cdot031\,48, \\ 1\cdot043\Delta M_0+3\cdot550\Delta m_0/m_0=-\cdot002\,70, \end{array}\right\} \qquad (57)$$

from which

$$\Delta M_0=-\cdot001\,371 \text{ (wt } 22\cdot38), \qquad (58)$$

$$\Delta m_0/m_0=-\cdot000\,357 \text{ (wt } 3\cdot50). \qquad (59)$$

A smooth ephemeris of "calculated" corrections, ΔM_c was prepared from (33), using these values, and was applied to the 31 normal values of $P-M-E$ to obtain the corrected ones. The sum of the squares of the latter was found to be $6\cdot047\times10^{-5}$ radians^2 ($=0\cdot1985$ deg.^{2}); the standard deviation is thus $\pm\cdot001\,444$ radians ($\pm0^\circ\cdot083$), and the (formal) probable errors of ΔM_0 and $\Delta m_0/m_0$ are $\pm\cdot000\,206$ (radians) and $\pm\cdot000\,520$ respectively. The former of these values, i.e. $\pm0'\cdot71$, indicates the accuracy with which the P.A. of the crescent at any stated time near mid-eclipse is determined by this method. Also, since

$$\Delta\left[\frac{d}{dt}\tan(M-M_0)\right]\simeq-\frac{n_0}{m_0}\frac{\Delta m_0}{m_0}$$

(n_0 being in effect not subject to correction), the probable error of the quantity $\dfrac{n_0}{m_0}\dfrac{\Delta m_0}{m_0}$, $\pm\cdot000282$, represents the uncertainty with which the rate of change of $\tan(M-M_0)$ is determined; near mid-eclipse this corresponds to an uncertainty in the rate of change of P.A. of $\pm0'\cdot97$/min.

Inserting (58) and (59), together with the above probable errors, in (36), and taking $x_0-\xi_0$ and $y_0-\eta_0$ from (50), we find

$$\left.\begin{array}{l} \Delta(x_0-\xi_0)=(-210\pm54)\times10^{-7}, \\ \Delta(y_0-\eta_0)=(+78\pm70)\times10^{-7}. \end{array}\right\} \qquad (60)$$

If we multiply these probable errors by the Earth's radius, we obtain ± 34 metres and ± 45 metres for the "geodetic" probable errors, parallel to the fundamental plane. The values on the Earth's slanting surface are of course larger, by the secants of the slants, but these will seldom exceed 2 or 2·5.

Although they are not actually required, we may readily obtain a new mid-eclipse time, and new mid-eclipse values of M, etc., if we correct the polynomials (50) by these amounts. Denoting the new values by the subscript 2, we find by the usual methods

$$\left.\begin{aligned} t_2 - t_0 &= +0^{\mathrm{m}}{\cdot}002\,54 \pm 0^{\mathrm{m}}{\cdot}000\,38, \\ t_2 &= 04^{\mathrm{h}}\,27^{\mathrm{m}}{\cdot}912\,77, \\ M_2 &= 34^{\circ}{\cdot}905, \\ N_2 &= 124^{\circ}{\cdot}905, \end{aligned}\right\} \tag{61}$$

thus directly confirming the practical invariability of the mid-eclipse value of M. The probable error of the time of mid-eclipse (± 23 milliseconds) makes it clear that systematic timing errors less than 5 ms or so would be quite unimportant; the time system actually used thus appears of fully adequate accuracy.

Adding the corrections (49) to (60), so as to reduce the results to the Almanac, we have

$$\left.\begin{aligned} \Delta(x_0 - \xi_0) &= (-284 \pm 54) \times 10^{-7}, \\ \Delta(y_0 - \eta_0) &= (-172 \pm 70) \times 10^{-7}, \end{aligned}\right\} \tag{62}$$

and the same values of t_2, M_2, N_2 will of course result if these are applied to the polynomials (46) and (47).

Differential corrections.—The basic data which may have to be changed, in order to produce the changes Δ, are as follows:

(*a*) The R.A. and Dec. of the Sun and Moon. The effects of changes in these on the tabular x and y are well known, and (approximate) formulae for them are quoted in the Almanacs. A change in the Sun's place does however introduce, in addition, a practically identical change in the coordinates, μ and d, of the apex, and this causes changes in ξ and η. We shall need to correct the Sun's place (on independent grounds), and we therefore include terms in $\Delta\mu_0$ and Δd_0 as well as in Δx_0 and Δy_0.

(*b*) The latitude, longitude, and height of the station will be "unknowns" in geodetic cases, but must be regarded as given in the present work. Corrections $\Delta\phi$, $\Delta\lambda$, and Δh are however included in the analysis, to allow for the possibility that the coordinates of Mombasa may have to be adjusted at some future date on independent grounds; incidentally, they indicate the effect (very small, but readily calculable) of polar variation at Mombasa.

(*c*) The Earth's ellipticity and the lunar parallax are related; the parallax with respect to the Earth's "mean" radius may be regarded (7) as not subject to correction, but for this reason the "equatorial" parallax is necessarily changed if the ratio of equatorial radius to mean radius is changed, i.e. if the ellipticity is. Brown used the relation given by Newcomb,

$$\pi_e = \pi_m(1 + \tfrac{1}{3}k), \tag{63}$$

where k is the ellipticity, to connect the equatorial parallax, π_e, with the mean one, π_m. The situation is a little unsatisfactory, since Brown's Tables are based on a parallax corresponding to the ellipticity $\frac{1}{294}$, while the S and C of the

Almanacs are computed for $\frac{1}{297}$; if $\frac{1}{297}$ is adopted (as it nominally is), Brown's parallax should be diminished by $0''\cdot04$, and the other consequential changes listed by him should also be made. It would be simpler, in the present case, to use S and C values based on $\frac{1}{294}$ and to keep the Almanac x and y values (**8**); the discrepancy between the two ellipticities was however not noticed until the analysis had been completed, and it has not been thought worth while to repeat it all with self-consistent data since the effect on the probable errors would clearly be negligible. The question is rather more complex than this discussion indicates (**9**), and until it has been fully clarified there seems little point in allowing for corrections to the parallax.

(*d*) The time system. A correction (Δt) to the entire time system is provided for, principally so that polar variation and the "annual" term in the Earth's rotation can be dealt with differentially. The correction is without effect on $t-t_0$ and thus does not affect the equation of condition; it is sufficient to include it at the present stage.

Accordingly, if we defer for the moment the explicit treatment of Δx_0 and Δy_0, we may re-write (62) as

$$\Delta x_0 - \frac{\partial \xi_0}{\partial \phi}\Delta\phi - \frac{\partial \xi_0}{\partial(\mu_0-\lambda)}\Delta(\mu_0-\lambda) - \frac{\partial \xi_0}{\partial h}\Delta h + (\dot{x}_0 - \dot{\xi}_0)\Delta t = (-284 \pm 54)\times 10^{-7},$$

$$\Delta y_0 - \frac{\partial \eta_0}{\partial \phi}\Delta\phi - \frac{\partial \eta_0}{\partial(\mu_0-\lambda)}\Delta(\mu_0-\lambda) - \frac{\partial \eta_0}{\partial d_0}\Delta d_0 - \frac{\partial \eta_0}{\partial h}\Delta h + (\dot{y}_0 - \dot{\eta}_0)\Delta t$$
$$= (-172 \pm 70)\times 10^{-7}. \tag{64}$$

From (6) and the standard formulae for $\rho\cos\phi'$ and $\rho\sin\phi'$ we have

$$\frac{\partial \xi_0}{\partial \phi} = -SC^2 \sin\phi \sin(\mu_0-\lambda) = -0\cdot065\,9, \tag{65}$$

$$\frac{\partial \xi_0}{\partial(\mu_0-\lambda)} = \xi_0 \cos d_0 - \eta_0 \sin d_0 = +0\cdot353\,3, \tag{66}$$

$$\frac{\partial \xi_0}{\partial h} \simeq \xi_0/6378 = -1\,462\times 10^{-7} \text{ if } \Delta h \text{ is in km}, \tag{67}$$

$$\frac{\partial \eta_0}{\partial \phi} = +SC^2[\cos\phi\cos d_0 + \sin\phi \sin d_0 \cos(\mu_0-\lambda)] = +0\cdot965\,9 \tag{68}$$

$$\frac{\partial \eta_0}{\partial(\mu_0-\lambda)} = \xi_0 \sin d_0 = +0\cdot231\,9, \tag{69}$$

$$\frac{\partial \eta_0}{\partial d_0} = -\zeta_0 = -0\cdot359\,8, \tag{70}$$

$$\frac{\partial \eta_0}{\partial h} \simeq \eta_0/6378 = +30\times 10^{-7}. \tag{71}$$

Thus if $\Delta\phi$ and $\Delta(\mu_0-\lambda)$ are understood to be in seconds of arc, and Δt in seconds of time,

$$10^7\Delta x = -284 \pm 54 - 3\cdot20\Delta\phi + 17\cdot13\Delta(\mu-\lambda) - 1462\Delta h - 1170\Delta t,$$
$$10^7\Delta y = -172 \pm 70 + 46\cdot83\Delta\phi + 11\cdot24\Delta(\mu-\lambda) - 17\cdot44\Delta d + 30\Delta h + 816\Delta t, \tag{72}$$

where we have now dropped the subscripts; to the accuracy we are using, these corrections refer to any fixed time near mid-eclipse, and are constant.

If we differentiate the formulae which define x and y, (**10**), treating $\pi_{☾}$ as constant and setting $\cos(\alpha_{☾}-a)\simeq\cos(\delta_{☾}-d)\simeq 1$, we obtain

$$\left.\begin{aligned}\sin\pi_{☾}\Delta x&=\cos\delta_{☾}(\Delta\alpha_{☾}-\Delta a)-\sin\delta_{☾}\sin(\alpha_{☾}-a)\Delta\delta_{☾},\\ \sin\pi_{☾}\Delta y&=\Delta\delta_{☾}-\Delta d+\cos\delta_{☾}\sin d\sin(\alpha_{☾}-a)(\Delta\alpha_{☾}-\Delta a).\end{aligned}\right\}\tag{73}$$

In these, we may write with sufficient accuracy,

$$\Delta\alpha_{☾}-\Delta a=\frac{1}{1-b}(\Delta\alpha_{☾}-\Delta\alpha_{\odot})=1\cdot0025(\Delta\alpha_{☾}-\Delta\alpha_{\odot})\tag{74}$$

and

$$\Delta\delta_{☾}-\Delta d=\frac{1}{1-b}(\Delta\delta_{☾}-\Delta\delta_{\odot})=1\cdot0025(\Delta\delta_{☾}-\Delta\delta_{\odot}),\tag{75}$$

and if $\Delta\alpha$ and $\Delta\delta$ are in seconds of arc we then have

$$\cos\delta_{☾}(\Delta\alpha_{☾}-\Delta\alpha_{\odot})-\cdot0041\Delta\delta_{☾}=3568''\Delta x,\tag{76}$$

$$\Delta\delta_{☾}-\Delta\delta_{\odot}+\cdot0041\cos\delta_{☾}(\Delta\alpha_{☾}-\Delta\alpha_{\odot})=3568''\Delta y,\tag{77}$$

or, solving explicitly for $\Delta\alpha_{☾}$ and $\Delta\delta_{☾}$,

$$\left.\begin{aligned}\cos\delta_{☾}(\Delta\alpha_{☾}-\Delta\alpha_{\odot})-\cdot0041\Delta\delta_{\odot}&=3568''(\Delta x+\cdot0041\Delta y),\\ \Delta\delta_{☾}-\Delta\delta_{\odot}&=3568''(\Delta y-\cdot0041\Delta x).\end{aligned}\right\}\tag{78}$$

Combing (78) with (72), and writing $\Delta\mu\simeq-\Delta\alpha_{\odot}$, $\Delta d\simeq+\Delta\delta_{\odot}$, we have as our experimental results

$$\left.\begin{aligned}\cos\delta_{☾}(\Delta\alpha_{☾}-0\cdot994\Delta\alpha_{\odot})-\cdot004\Delta\delta_{\odot}&=-0''\cdot102\pm0''\cdot019\\ &\quad-\cdot0010\Delta\phi-\cdot0061\Delta\lambda-\cdot522\Delta h-\cdot417\Delta t,\\ \Delta\delta_{☾}-0\cdot994\Delta\delta_{\odot}+\cdot004\Delta\alpha_{\odot}&=-0''\cdot061\pm0''\cdot025\\ &\quad+\cdot0167\Delta\phi-\cdot0040\Delta\lambda+\cdot013\Delta h+\cdot293\Delta t,\end{aligned}\right\}\tag{79}$$

where, as already stated, $\Delta\phi$ and $\Delta\lambda$ are in seconds of arc, Δh is in km, and Δt is in seconds of time.

Polar variation.—The international coordinates of the pole, interpolated to 1948·835, are $-0''\cdot032$, $+0''\cdot196$. These introduce corrections to the coordinates of the station of $\Delta_p\phi=-0''\cdot15$, $\Delta_p\lambda=+0''\cdot01$, together with a time correction $\Delta_p t=-16\cdot4$ ms, since the time used was that determined at Greenwich. Accordingly, we must add to the right-hand side of (79)

$$\left.\begin{aligned}\cos\delta_{☾}\Delta_p\alpha_{☾}&=+0''\cdot007,\\ \Delta_p\delta_{☾}&=-0''\cdot007.\end{aligned}\right\}\tag{80}$$

Annual variation.—This can be treated in a similar way, i.e. by a time change combined with a longitude change, but in one sense it is a matter of opinion whether one actually should apply this correction or not, since the annual variation is only one part of the complete E.T.–U.T. difference, while it is the whole difference which is connected with $\Delta\alpha_{☾}$. Since, however, occultation results show a number of oscillations which are not yet fully explained, so that nothing less than an annual mean can be relied on for estimating E.T.–U.T. from them, it appears best to remove the annual variation, as determined by clocks, in the present case. Indeed, the application to lunar observations

(whatever the method) of an independently determined correction for short-period E.T.–U.T. fluctuations may still remain for some years preferable to any attempt to infer those fluctuations from the lunar observations directly; lunar observations by themselves are not very well suited for separating E.T.–U.T. fluctuations from errors in the Moon's elements.

If the correction to be added to the time system used, in order to reduce it to "uniform" time for the year in question, is $\Delta_e t$ (seconds), we must insert this, and also the longitude correction $\Delta_e\lambda = +15''\Delta_e t$, in (79). $\Delta_e t$ for 1948 is not particularly well determined; we adopt -58 ms for November 1 (from Finch (**11**), using the 1948–9 figures only and ignoring any $\Delta_a\alpha$ term that may affect the FK3 places), and so

$$\left.\begin{aligned} \cos\delta_{☾}\Delta_e\alpha_{☾} &= +0''\cdot030, \\ \Delta_e\delta_{☾} &= -0''\cdot014, \end{aligned}\right\} \tag{81}$$

must be added to the right-hand side of (79).

The equinox.—Eclipse observations refer the Moon to the true Sun, whose place is rigorously computed with respect to the true equinox. The present observations, however, have used a time system based on the FK3 equinox. The formula for "Sidereal Time of 0^h U.T." is not itself affected by an equinox change, but the experimental determinations of U.T. are: if the R.A.'s of the FK3 stars are too small by a quantity E (seconds of time), the U.T. employed in the observations will require a correction $\Delta_E t = +E$. The Washington observations probably afford the best recent determinations of E: the value adopted from the 9-inch T.C. observations 1935–45 (**12**) is $E = +0^s\cdot022$, with a probable error (based on the discrepancies and weights) which is readily found to be $\pm 0^s\cdot004\,3$; the value adopted from the 6-inch T.C. observations 1941–49 (**13**) was $+0^s\cdot023$ with a p.e. which may similarly be obtained as $\pm 0^s\cdot005\,1$. We adopt, therefore, $\Delta_E t = +0^s\cdot022 \pm 0^s\cdot003$ (**14**), so that the quantities

$$\left.\begin{aligned} \cos\delta_{☾}\Delta_E\alpha_{☾} &= -0''\cdot009 \pm 0''\cdot001, \\ \Delta_E\delta_{☾} &= +0''\cdot006 \pm 0''\cdot001, \end{aligned}\right\} \tag{82}$$

must be added to the right-hand sides of (79).

Light-time and aberration.—Brown's Tables take no explicit notice of aberration at all, and the Moon's place in the Almanacs is computed without applying it; this caused some difficulty at the start of the present work. The question was raised with Dr G. M. Clemence, who replied that he was already engaged on a review of several related matters and would include this one. The point has since been fully clarified (**15**); almost all of the effects of aberration are taken up (in the tables) by a slight falsification of the lunar elements, but there remain certain small correction-terms which should be applied. The result should then strictly be the Moon's geocentric place; however, all observations (including those on which Brown based his tables) are also falsified by the fact that the light-time to the observer is less than that to the centre of the Earth. The difference amounts to $21\cdot3\zeta$ milliseconds, or $7\cdot6$ ms in the present case. The need for this correction appears to have been first pointed out by Banachiewicz (**16**); it is insignificant unless ζ is large, but we insert it here along with the other small terms. We have $\Delta_L t = +7\cdot6$ ms, $\Delta_L\lambda = +0''\cdot114$ and so

$$\left.\begin{aligned} \cos\delta_{☾}\Delta_L\alpha_{☾} &= -0''\cdot004, \\ \Delta_L\delta_{☾} &= +0''\cdot002. \end{aligned}\right\} \tag{83}$$

The additional corrections (**15**), to deal fully with the ordinary aberration, and the correction pointed out by Woolard (**17**) are omitted here, because the final result is expressed as a departure of the observed place of the Moon from that of Brown's Tables as they stand; for 1948, they do of course still contain even the Great Empirical Term, and the tabular places based on them are not in any case really adequate for geodetic work.

The Sun's place.—It does not appear to be generally realized that a correction to the Sun's mean longitude (which is applied in the Almanacs) is not the only correction which may be important. Corrections to the obliquity, eccentricity, and longitude of perihelion all produce changes in the Sun's R.A. and Dec., which at some times of the year can add up to very appreciable amounts, and these corrections seem relatively well known. For simplicity, and because the Cape results are more self-consistent than any others, we take the Cape corrections alone; the figures can readily be modified if an alternative choice were preferred. For the four periods 1907–11, 1912–16, 1925–32, and 1932–36, the corrections Δh and Δk $(h = e\cos\pi,\ k = e\sin\pi)$ have weighted means $\Delta h = -0''{\cdot}141 \pm 0''{\cdot}010$ and $\Delta k = +0''{\cdot}136 \pm 0''{\cdot}010$. A correction to the obliquity need not be applied; it affects the Moon's tabular place as much as the Sun's, and although the left-hand sides of (79) are not exactly $(\Delta_{☾} - \Delta_{\odot})$, they are sufficiently near this to make any admissible value of the correction cancel out. The corrections to the Sun's place, in addition to the correction $\Delta L_{\odot}$ which is already incorporated in the Almanacs (and which there seems no good reason to modify) are

$$\left.\begin{aligned} \Delta\alpha_{\odot} &= +2\sin\alpha\sec\delta\Delta h - 2\cos\alpha\sec\delta\cos\epsilon\Delta k, \\ \Delta\delta_{\odot} &= +2\cos\alpha\sin\delta\Delta h - 2\cos^2\alpha\cos\delta\sin\epsilon\Delta k, \end{aligned}\right\} \qquad (84)$$

or for 1948 November 1·186,

$$\left.\begin{aligned} \Delta\alpha_{\odot} &= +0''{\cdot}380 \pm 0''{\cdot}021, \\ \Delta\delta_{\odot} &= -0''{\cdot}125 \pm 0''{\cdot}007, \end{aligned}\right\} \qquad (85)$$

and these should also be inserted in (79). By comparison, we may note that the corrections already included, on account of $\Delta L_{\odot}$, are

$$\left.\begin{aligned} \Delta_L\alpha_{\odot} &= +1''{\cdot}00 \pm 0''{\cdot}060, \\ \Delta_L\delta_{\odot} &= -0''{\cdot}33 \pm 0''{\cdot}020. \end{aligned}\right\} \qquad (86)$$

The probable errors in (86) are based on a p.e. for $\Delta L_{\odot}$ of $\pm 0''{\cdot}060$; this value depends on questions of weighting which are to some extent a matter of opinion, but it seems that the values in (86) are at any rate considerably larger than those in (85).

The Moon's place.—Collecting eqns. (79), (80), (81), (82), (83), and (85) we obtain

$$\left.\begin{aligned} \cos\delta_{☾}\Delta\alpha_{☾} &= +0''{\cdot}288 \pm 0''{\cdot}019 - 0{\cdot}0010\Delta\phi - 0{\cdot}0061\Delta\lambda - 0{\cdot}522\Delta h, \\ \Delta\delta_{☾} &= -0''{\cdot}200 \pm 0''{\cdot}025 + 0{\cdot}0167\Delta\phi - 0{\cdot}0040\Delta\lambda + 0{\cdot}013\Delta h, \end{aligned}\right\} \qquad (87)$$

where we have assumed that no further corrections to the Sun's place, or to the time system, will be required, but have still kept the terms which would be needed in geodetic work; we have also, for the moment, omitted the probable errors of the Sun's place, since these are irrelevant in geodetic applications. (The probable errors of all the other corrections will ordinarily be too small to have any effect, by comparison with 0″·02.) In order to express this result in the

form of corrections to Brown's Tables, we must now add the special eclipse corrections that were applied (by Mrs Gossner and C.A.M., in this case) when computing the Besselian elements. These were $-1''\cdot720$ in $\alpha_{☾}$ (i.e. $-1''\cdot666$ in $\alpha_{☾}\cos\delta_{☾}$) and $+0''\cdot190$ in $\delta_{☾}$. Accordingly

$$\left.\begin{aligned}\cos\delta_{☾}\Delta_B\alpha_{☾} &= -1''\cdot378 \pm 0''\cdot019 - 0\cdot0010\Delta\phi - 0\cdot0061\Delta\lambda - 0\cdot522\Delta h,\\ \Delta_B\delta_{☾} &= -0''\cdot010 \pm 0''\cdot025 + 0\cdot0167\Delta\phi - 0\cdot0040\Delta\lambda + 0\cdot013\Delta h,\end{aligned}\right\} \quad (88)$$

are the corrections to Brown's Tables demanded by the present results.

We now transform these into orbital components. At $04^h\,28^m$, the hourly rates of change of $\alpha_{☾}$ and $\delta_{☾}$ were $+135^s\cdot84$ and $-888''\cdot0$, while $\delta_{☾}$ was $-14°\,21'\cdot6$; the P.A. of the Moon's (geocentric) orbit was thus $90°+24°13'$, and its orbital speed was $2\,164''\cdot5$/hr. Rotating axes in (88) through $24°13'$ we have the orbital corrections

$$\Delta_B L = -1''\cdot253 \pm 0''\cdot020 - 0\cdot008\Delta\phi - 0\cdot004\Delta\lambda - 0\cdot481\Delta h, \quad (89)$$

$$\Delta_B B = -0''\cdot574 \pm 0''\cdot024 + 0\cdot015\Delta\phi - 0\cdot006\Delta\lambda - 0\cdot202\Delta h, \quad (90)$$

and multiplying (89) by $1976''\cdot5/2\,164''\cdot5$ we obtain the mean longitude correction as

$$\Delta_B L_m = -1''\cdot144 \pm 0''\cdot018 - 0\cdot007\Delta\phi - 0\cdot004\Delta\lambda - 0\cdot439\Delta h. \quad (91)$$

The coefficients of the correction terms can of course be justifiably carried to several more places, if $\Delta\phi$ and $\Delta\lambda$ seem likely to be large.

In geodetic work, a pair of equations of the form (90) and (91) would be obtained for each station, and the same values of $\Delta_B B$ and $\Delta_B L_m$ would then be adopted for all, probably by assuming that the values of $\Delta\phi$, $\Delta\lambda$, and Δh for some one station were zero, or, more generally, by weighting all the absolute terms according to the supposed certainties of their geographical coordinates. Unfortunately, it seems that some arbitrary decision of this sort would always have to be made; for even if it is assumed that Δh is zero for all stations, there are still $2n+2$ unknowns for n stations, namely ΔL_m and ΔB in addition to the two coordinates of each station, and there are only $2n$ equations. It thus appears that even in order to get a geodetic result it is necessary to derive (or assume) *absolute* corrections to the Moon's place (and the Sun's); it is not, for example, true that changing the longitudes of two stations by equal amounts would alter the absolute terms of all the eqns. (90), or (91), by the same amounts, though such a change would leave the geodetic separation of the stations unaffected.

Probable errors.—The probable errors of (90) and (91) are those which apply in geodetic work. In many respects it is also these which are most directly comparable with those obtained by other methods. It is true that in the present case the p.e. of ΔM_0 should be increased, on account of the uncertainty in the zero of position-angles; the p.e. there (2) was $\pm 0°\cdot022 = \pm\cdot000\,384$ radians, and neglecting the fact that there is a varying weighting, $\sec^2(M-M_0)$, involved, roughly this figure should be combined with the $\pm\cdot000\,206$ quoted for ΔM_0. In future cases, with the scatter of Fig. 3 much reduced by the use of fiducial marks, and with three or four times as long a diurnal run to rely on if the bench is properly stabilized, this contribution should be much reduced; it may, however, always remain a factor to be considered. The p.e. of $\Delta m_0/m_0$ ($\pm\cdot000\,520$) is practically unaffected by uncertainty in the P.A. zero, and those of $\Delta(x-\xi)$ and $\Delta(y-\eta)$ are thus not greatly affected by it, even in the present case.

A more serious objection is that the residuals $P-M-E$ still show some appreciable systematic features, so that the formulae for the probable errors (and indeed the least-squares process itself) are of doubtful validity. The 310 values of $P-M$ plotted in Fig. 9 are those which result from the least-squares solution, i.e. the original $P-M$ have been corrected by the smooth ephemeris ((33), (58), (59)) already mentioned; it is evident that in addition to several short stretches there is one longer one, at the end, where they depart systematically from the E values. The least-squares process is valid only on the assumption that successive residuals are independent, and this is not so here.

The question how to obtain a "probable error" in such cases is peculiarly liable to arise in work involving the Moon's limb. If, for example, the coordinates of the centre of an observed arc of lunar limb are deduced by a least-squares process from the lengths of a number of radii measured from a provisional centre, the probable errors of the result are sometimes quoted without any special discussion of possible systematic runs in the residuals. Obviously if the number of radii is made large enough for many to fall within the same "mountain" or "valley", systematic trends will appear; the number of radii used will then have an appreciable effect on the formal probable errors obtained, even though it has very little on the coordinates of the centre themselves, and if too many are used the formal probable errors may fall considerably below the true ones. In general, it is for this reason unprofitable to use more than 15 or 20 radii for a lunar semi-circumference; and since the mountains and valleys can run to $\pm 2''$, the probable errors of the Moon's place at any one instant cannot be less than $0''\cdot 1$ or $0''\cdot 2$ whatever the method of observation, so long as limb corrections are not applied.

When they are applied, of course, the true probable errors can be much reduced; on the one hand, the residuals themselves are much reduced, and on the other, their independence of each other should persist even if many more are used. The question how many one can use is not altogether self-evident, but the following consideration can provide some guidance (**18**). If w residuals were all completely independent, the number of changes of sign if they were run through in order would be $w/2$ on the average; it is thus reasonable to consider that the number of residuals which can fairly be regarded as independent is not greater than twice the number of their changes of sign. However, this condition, though it appears to be necessary, is not sufficient; the 310 residuals of Fig. 9 change sign about 87 times, which suggests a fairly high degree of independence, but the true extent is much less, owing to the way the changes are distributed, and there can be no doubt that the uncertainty of the final result is greater than is suggested by the probable errors obtained, even though these have only assumed that the 31 normal points are independent.

The residual systematic effects which are still present in Fig. 9 may be partly due to occasional slow waves of atmospheric distortion; they are perhaps due in part to instability of the bench, of some other kind than that already taken into account (and to this extent it seems likely that they can be eliminated in future work); they may to a small extent be due to the fact that the librations of the Washington traces were not exactly those of the eclipse (though such comparisons as could be made suggested that these discrepancies were not very serious); and they may also be due to the fact that Dr Watts has still some further systematic baseline revisions to carry out. These could not account for any of the short

systematic divergences in Fig. 9, but they might in principle almost entirely remove such a trend as is shown by the final 42 points.

The absolute probable errors of the Moon's place are greater than the "geodetic" ones of (89)–(91), owing to the rather large uncertainty which still attaches to the Sun's corrections. For this purpose we must include the probable errors of both (85) and (86); these combine to $\pm 0''\cdot 067$ in ecliptic longitude and, of course, $\pm 0''\cdot 000$ in ecliptic latitude, and rotating axes to the direction of the Moon's orbit we get $\pm 0''\cdot 067$ in orbital longitude and $\pm 0''\cdot 006$ in latitude. Accordingly we may write (89), (90), and (91) as

$$\Delta L = -1''\cdot 25 \pm 0''\cdot 070, \tag{92}$$

$$\Delta L_m = -1''\cdot 14 \pm 0''\cdot 064, \tag{93}$$

$$\Delta B = -0''\cdot 57 \pm 0''\cdot 025, \tag{94}$$

together with the same terms in $\Delta\phi$, $\Delta\lambda$, and Δh as before, if these are needed.

It is these results which should be most nearly comparable with those obtained from occultations. Even these, however, cannot be directly compared with either the lunation means or the quarterly ones. If they are compared with the annual ones, they should in principle give the effect, at the time of the eclipse, of those errors in the tables whose period is comparable with a year, or is much shorter. As they stand, however, the occultation results have not been corrected for polar variation or the FK3 equinox.

The largest single uncertainty, so far as the above probable errors are concerned, is that in the Sun's mean longitude; until this is decreased, the accuracy with which the Moon's place can be determined from eclipses cannot now be much reduced by any further experimental advances. In the present case there are undoubtedly still some systematic errors also; in particular, there is the uncertainty in the zero of position-angles (though the good agreement with the mean of Watts' systems suggests that there is little actual error here), and the fact that the crescents were only measured in one orientation of the film. Further, Dr Watts' baselines are not yet final, and the geographical coordinates of Mombasa may also need some revision. If these uncertainties can be eliminated in future work (as seems indeed quite possible), the probable errors in (90) and (91) should indicate the geodetic reliability to be obtained by this method.

Royal Observatory,
Greenwich :
1954 *November* 30.

Royal Greenwich Observatory,
Herstmonceux :
1954 *November* 30.

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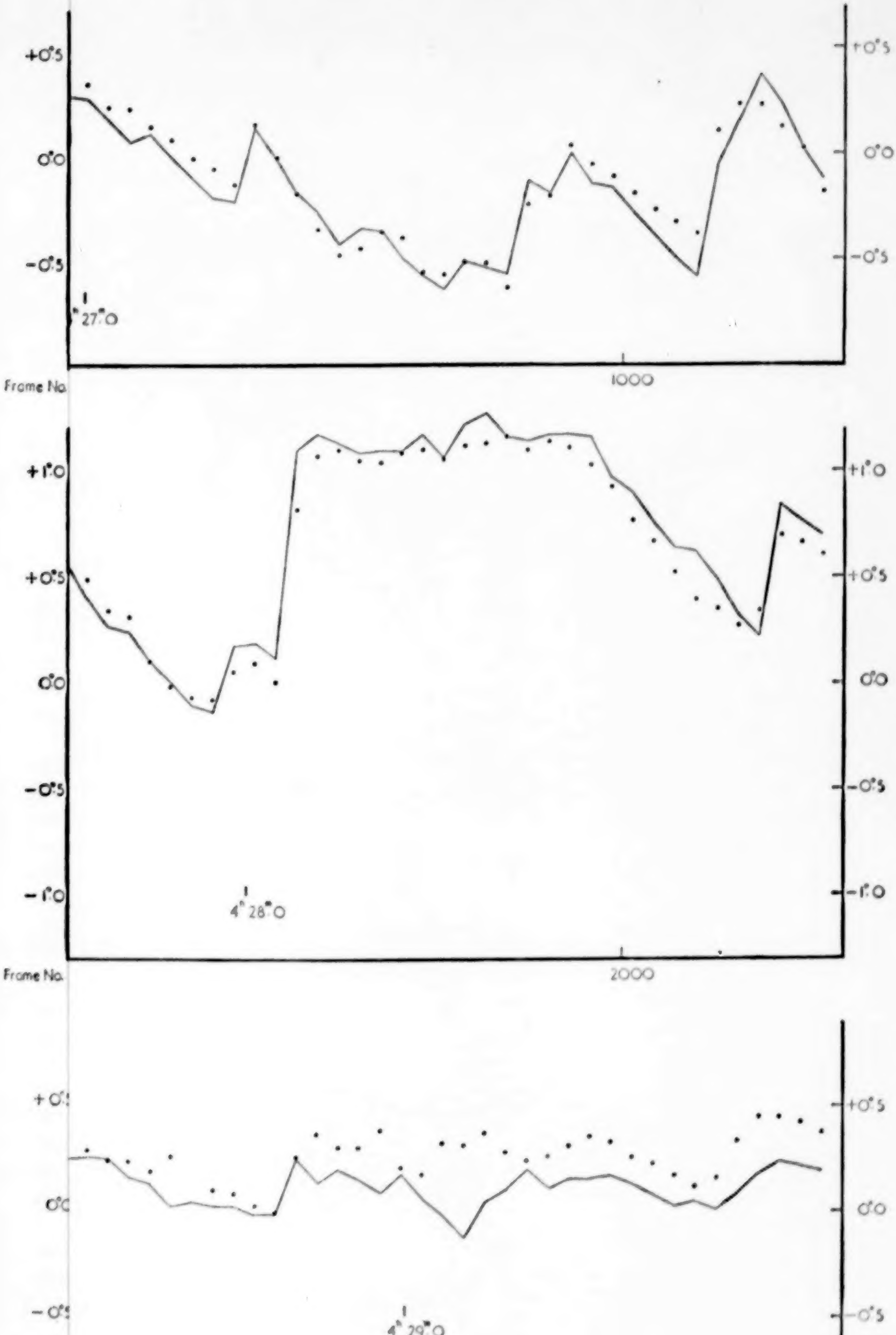
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+0°5
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4h 28m0
Frame No.
2000
+0°5
0°0
−0°5
4h 29m0
Frame No.
3000

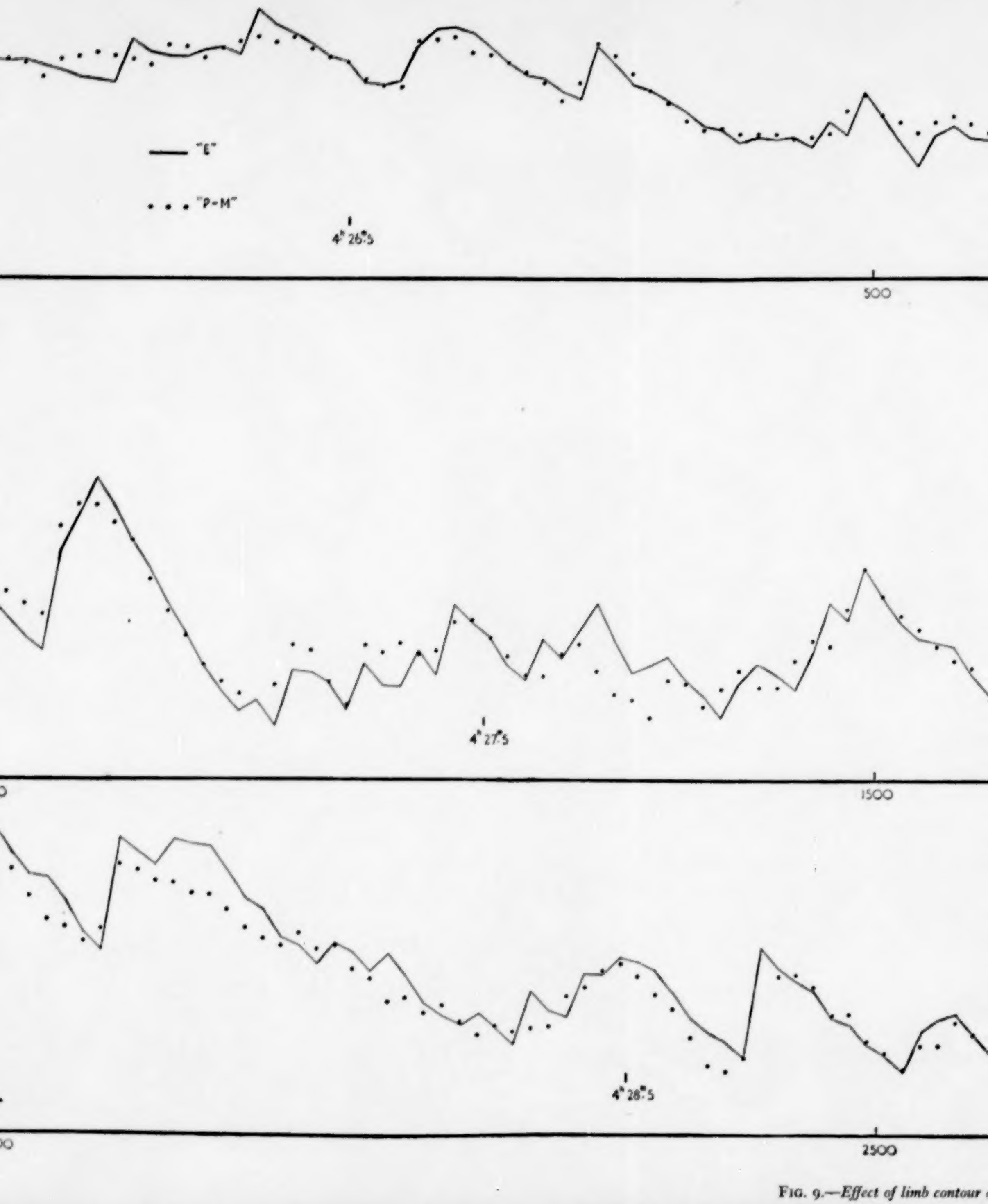
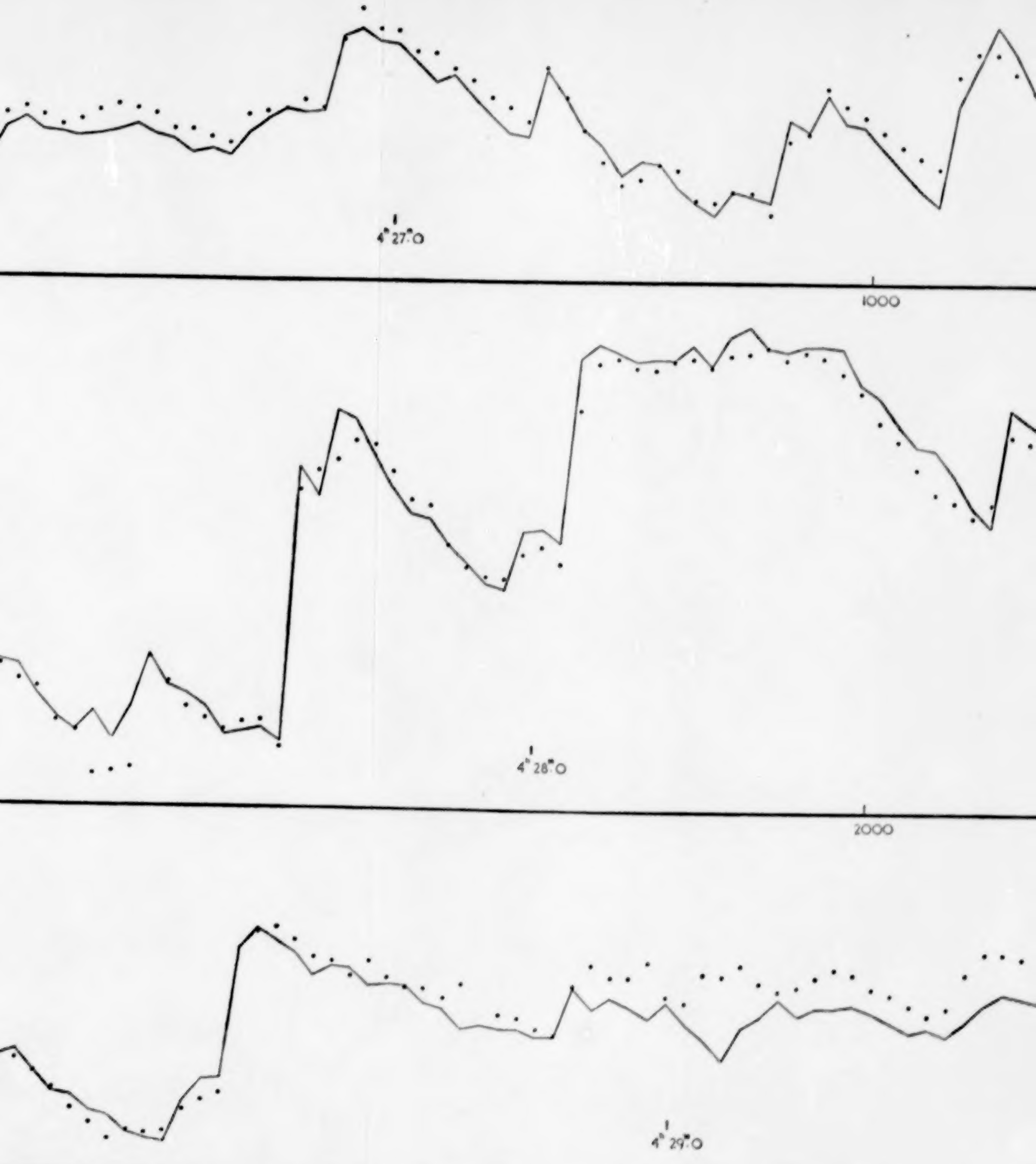


FIG. 9.—*Effect of limb contour*



ontour on position-angles.

APSIDAL-MOTION CONSTANTS FOR POLYTROPIC MODELS

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Summary

Apsidal-motion constants for the polytropic family of models have been integrated with the aid of the Manchester University Electronic Computer Mk II for polytropic indices $n = 1(0{\cdot}5)3{\cdot}0(0{\cdot}25)4{\cdot}75$ and jth harmonic distortion, $j = 2(1)7$. In order to do so, Clairaut's and Emden's equations have been integrated simultaneously, with an accuracy of seven decimal places. The results of this work are summarized, and compared with less accurate computations of previous investigations.

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1. *Introduction.*—The determination of apsidal-motion constants which specify the rate of apsidal advance arising from jth harmonic distortion in close binary systems has been recognized for some time as the unique way for gaining empirical insight into the internal structure of the constituent components; and a considerable effort has been spent in determining the theoretical values of such constants appropriate for different stellar models.

Following an early work by Russell (**1**) in which mainly configurations exhibiting a low degree of central condensation were considered, extensive integrations leading to the determination of apsidal-motion constants were undertaken by Chandrasekhar (**2**) for the polytropic family of models, and more recently by Motz (**3**) and Keller (**4**) for certain other models. Of all such models, those belonging to the polytropic family have so far been investigated in this respect more extensively than any other group, and the results made the basis of several discussions (**5, 7**). Unfortunately, the accuracy of numerical results on the basis of such investigations leaves, as yet, something to be desired. For instance, Chandrasekhar's value of the apsidal-motion constant corresponding to a model with a polytropic index $n = 3{\cdot}5$, $j = 2$ has been reported inaccurate, and corrected by him several years ago (**6**), and quite recently his value for $n = 4$, $j = 2$ was likewise challenged (**7**) as being too large by 12 per cent. In view of this uncertainty it was thought desirable to redetermine apsidal-motion constants for polytropic models characterized by $n = 1(0{\cdot}5)3{\cdot}0(0{\cdot}25)4{\cdot}75$ invoked by spherical harmonic distortion of order $j = 2(1)7$, by methods which should leave no room for doubt as to the degree of accuracy attained, and provide a reliable basis for any subsequent discussion.

2. *Statement of the problem.*—As is well known (**8**), the constants k_j of apsidal motion, in eccentric binary systems, invoked by the jth harmonic distortion of their components of arbitrary structure, are defined by

$$k_j = \frac{j + 1 - \eta_j(a_1)}{2j + 2\eta_j(a_1)}, \tag{1}$$

where $\eta_j(a_1)$ denotes the surface value of a particular solution of the Clairaut-Radau differential equation

$$a\frac{d\eta_j}{da}+\eta_j(\eta_j-1)+6D(\eta_j+1)=j(j+1), \tag{2}$$

subject to the initial condition

$$\eta_j(0)=j-2. \tag{3}$$

Here $D\equiv\rho/\bar{\rho}$, where $\rho(a)$ denotes the density of the respective undistorted configuration at a distance a from its centre, while $\bar{\rho}(a)$ stands for the mean density inside the volume of radius r.

For the polytropic family of models,

$$\rho=\rho_c y^n, \tag{4}$$

where ρ_c denotes the central value of ρ, and y stands for the Lane–Emden function satisfying the differential equation

$$\frac{1}{a^2}\frac{d}{da}\left(a^2\frac{dy}{da}\right)+y^n=0, \tag{5}$$

subject to the initial conditions

$$y(0)=1 \qquad \text{and} \qquad y'(0)=0. \tag{6}$$

In consequence,

$$D=-\frac{1}{3}\frac{ay^n}{y'}=\frac{1}{3}U,$$

where U denotes a function which has already been tabulated (**9**).

The British Association Tables, which have formed the basis of Chandrasekhar's work, give U to 5 significant figures—an accuracy which is generally quite insufficient to guarantee much more than 2 significant figures for computed apsidal-motion constants, and which may partly account for the lack of consistency in Chandrasekhar's results. This difficulty does not arise when using an automatic computing machine because it is more convenient to incorporate the Lane–Emden equation with the Clairaut–Radau equations and to integrate the complete system, i.e.

$$a\frac{d\eta_j}{da}+\eta_j(\eta_j-1)+6D(\eta_j+1)=j(j+1), \quad \text{for} \quad j=2(1)7, \tag{7}$$

and

$$\frac{d^2y}{da^2}+\frac{2}{a}\frac{dy}{da}+y^n=0, \tag{8}$$

subject to the initial conditions

$$\eta_j(0)=j-2,$$
$$y(0)=1,$$
$$y'(0)=0.$$

3. *Method of solution.*—The numerical solution of the joint system of equations (7) and (8) was evaluated on the Manchester University Electronic Computer, Mk II. It was decided to use the Runge–Kutta method as described in (**10**), which is a standard process on most electronic machines. The method is usually applied to a set of simultaneous first-order equations in which the

independent variable does not appear explicitly. To arrange the system in this form, Emden's equation is first transformed to a pair of simultaneous first-order equations by introducing the new variable

$$v = -\frac{1}{a}\frac{dy}{da}. \tag{9}$$

With this substitution (8) is then replaced by the equations

$$\frac{dy}{da} = -av, \tag{10}$$

$$a\frac{dv}{da} = y^n - 3v, \tag{11}$$

subject to the initial conditions $y(0) = 1$, $v(0) = \frac{1}{3}$; the latter being obtained from the expansion

$$y = 1 - \frac{1}{3!}a^2 + \frac{n}{5!}a^4 - \frac{(8n^2 - 5n)}{3 \cdot 7!}a^6 + \frac{(122n^3 - 183n^2 + 70n)}{9 \cdot 9!}a^8 - \dots$$

which holds in the neighbourhood of the origin.

The equations can be integrated directly, but the coefficient a^{-1} causes a scale factor difficulty and necessitates using a short interval near the origin. For this reason it is convenient to make the transformation

$$x = \ln a, \quad \frac{d}{dx} = a\frac{d}{da},$$

as a result of which the equations (7) take the form

$$\frac{d\eta_j}{dx} = j(j+1) - 6D(\eta_j + 1) - \eta_j(\eta_j - 1), \quad j = 2(1)7. \tag{12}$$

For reasons of uniformity the equations (10) and (11) are transformed similarly, obtaining

$$\frac{dv}{dx} = y^n - 3v, \tag{13}$$

$$\frac{dy}{dx} = -a^2 v. \tag{14}$$

The set is completed by including the equation

$$\frac{da}{dx} = a, \tag{15}$$

which connects a and x.

Adopting a more systematic notation the nine equations (12), (13), (14) and (15) are written

$$\frac{dy_1}{dx} = y_1,$$

$$\frac{dy_2}{dx} = -3y_2 + y_3^n,$$

$$\frac{dy_3}{dx} = -y_1^2 y_2,$$

$$\frac{dy_4}{dx} = 6 - y_4(y_4 - 1) - 6D(y_4 + 1).$$

$$\frac{dy_5}{dx}=12-y_5(y_5-1)-6D(y_5+1),$$

$$\frac{dy_6}{dx}=20-y_6(y_6-1)-6D(y_6+1),$$

$$\frac{dy_7}{dx}=30-y_7(y_7-1)-6D(y_7+1),$$

$$\frac{dy_8}{dx}=42-y_8(y_8-1)-6D(y_8+1),$$

$$\frac{dy_9}{dx}=56-y_9(y_9-1)-6D(y_9+1),$$

where

$$D=\frac{1}{3}\frac{y_3''}{y_2}.$$

The initial conditions at $a=0$, namely $y_1=0$, $y_2=\frac{1}{3}$, $y_3=1$, $y_4=0$, $y_5=1$, $y_6=2$, $y_7=3$, $y_8=4$, $y_9=5$, now correspond to $x=-\infty$.

In order to start the integration, it is necessary to assume initial conditions at a small but finite value of $y_1=a$, otherwise the integration of the y_1 equation, and hence the whole system, cannot proceed. Once started, however, the value of y_1 grows exponentially with a fixed relative accuracy. It is desirable to start with the largest value, a_0, of y_1 which allows the corresponding y_i to be evaluated from their expansions about the origin. These are

$$y_2=\frac{1}{3}\left(1-\frac{n}{10}a^2+\dots\right),$$

$$y_3=1-\frac{1}{3!}a^2+\frac{n}{5!}a^4-\dots,$$

$$D=1-\frac{n}{15}a^2+\dots,$$

and

$$\eta_j=(j-2)+\frac{2n}{5}\frac{(j-1)}{(2j+3)}a^2-\dots.$$

In fact, by taking $a_0=10^{-5}$ the values of these quantities agree with the true initial values to within 10^{-9}.

A consequence of the logarithmic transformation is that the optimum interval of integration is not the same at all parts of the range. At the "origin", where quantities are varying relatively slowly, a large interval can be used, which has then to be reduced steadily as the integration proceeds. This adjustment is carried out automatically by the programme in accordance with the following rules. Every few steps (the exact number varied between 3 and 10) the truncation error was tested by comparing the results of a single step of length h with that of two steps of length $h/2$. If the results agreed to within the required accuracy, then the integration is continued with the original step-length h; if not, then the interval was halved. In this way the truncation error for each step is reduced to less than 10^{-9}. The cumulative effect of the errors completed at each step is determined by examining the quantity $h\frac{\partial}{\partial y_i}\left(\frac{dy_i}{dx}\right)$, $i=4, 5, \dots, 9$. If this expression is negative, then neighbouring solutions converge and any errors

introduced tend to diminish. This condition holds everywhere in the range for the solutions considered below. These conclusions were confirmed in many cases by repeating the integrations with slightly different, $O(10^{-9})$, initial conditions. The results of such repetitions indicate that the last one or two figures are not reliable and for this reason the results (η_j) are given to 7 decimals only.

The values of η_j (i.e. $y_4, \ldots, y_9$) on the surface of the configuration are defined by the point at which y_3 (y in Emden's original equation) vanishes. This point is found by a Weierstrassian subdivision process as follows. The integration is carried forward until a negative value of y is encountered. The interval in which the root is isolated is then successively halved by advancing and retarding the integration until the required accuracy is obtained—n halvings yielding n binary places in the value of y.

4. *Results.*—Table I gives the values of η_j for values of $j = 2(1)7$ and values of $n = 1(0{\cdot}5)3(0{\cdot}25)4{\cdot}75$, together with $n = 0$ and $n = 5$ for completeness. Table II gives the corresponding apsidal-motion constants for these values of j and n; and this illustrates how the k's decrease with increasing n, and to a lesser extent

TABLE I

Values of η_j

n \ j	2	3	4	5	6	7
0	0·0	1·0	2·0	3·0	4·0	5·0
1	1·2898727	2·7712558	4·0322897	5·1985475	6·3146419	7·4006468
1·5	1·8863372	3·3308438	4·5325361	5·6471720	6·7205668	7·7711841
2	2·3558682	3·6743708	4·7975208	5·8605512	6·8976928	7·9215832
2·5	2·6741873	3·8601625	4·9225255	5·9513727	6·9670118	7·9763746
3	2·8596251	3·9485820	4·9746967	5·9885823	6·9909869	7·9939855
3·25	2·9145691	3·9711040	4·9866323	5·9927584	6·9956613	7·9972088
3·5	2·9512885	3·9848142	4·9934119	5·9966130	6·9980576	7·9987963
3·75	2·9744672	3·9926882	4·9970328	5·9985530	6·9992072	7·9995284
4	2·9880797	3·9968781	4·9988199	5·9994568	6·9997157	7·9998370
4·25	2·9953218	3·9988870	4·9996095	5·9998305	6·9999154	7·9999534
4·5	2·9986395	3·9997085	4·9999050	5·9999619	6·9999832	7·9999896
4·75	2·9998017	3·9999628	4·9999885	5·9999944	6·9999950	7·9999955
5	3·0	4·0	5·0	6·0	7·0	8·0

TABLE II

Values of k_j

n \ j	2	3	4	5	6	7
0	0·75000000	0·37500000	0·25000000	0·18750000	0·15000000	0·12500000
1	0·25990728	0·10645380	0·06023876	0·03929248	0·02782696	0·02080994
1·5	0·14327923	0·05284890	0·02739302	0·01656909	0·01098352	0·00774535
2	0·07393839	0·02439400	0·01150774	0·00641997	0·00396610	0·00262763
2·5	0·03485234	0·01019200	0·00434151	0·00222015	0·00127200	0·00078876
3	0·01444298	0·00369989	0·00140970	0·00051953	0·00034690	0·00020056
3·25	0·00869160	0·00207256	0·00074375	0·00032938	0·00016693	0·00009306
3·5	0·00491907	0·00108706	0·00036627	0·00015400	0·00007472	0·00004013
3·75	0·00256639	0·00052282	0·00016490	0·00006578	0·00003049	0·00001572
4	0·00119488	0·00022309	0·00006557	0·00002469	0·00001093	0·00000543
4·25	0·00046826	0·00007951	0·00002170	0·00000770	0·00000325	0·00000155
4·5	0·00013609	0·00002082	0·00000528	0·00000173	0·00000065	0·00000035
4·75	0·00001983	0·00000266	0·00000064	0·00000025	0·00000019	0·00000015
5	0	0	0	0	0	0

8

with increasing j. In fact it can be shown that $k \to 0$ as $j \to \infty$. A comparison of the values of these k's with those which may be obtained from Chandrasekhar's values (2)* will show agreement to 2 significant figures only.

In conclusion, it may be emphasized that the theory underlying Clairaut's equation assumes the distortion to remain so small that quantities of the order of the squares and cross-products of the individual harmonics may be neglected. If the outcome of such a theory is then applied to close binary systems, a retention of first-order terms can be shown to be tantamount to regarding the disturbing action of the secondary component as that of a mass-point. In order to remove this limitation, Clairaut's equation must be augmented by the retention of second-order terms. An investigation aiming to do so is now in progress, and it is hoped to publish its results in the near future.

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* In Chandrasekhar's work (2) a somewhat different system of notation is used. The relationship connecting his notation and that used in this paper is

$$k_j = \frac{1}{2}\left\{ \frac{j - \left(\dfrac{\xi_1 \psi_j'(\xi_1)}{\psi_j(\xi_1)} \right)}{j + 1 + \left(\dfrac{\xi_1 \psi_j'(\xi_1)}{\psi_j(\xi_1)} \right)} \right\}.$$

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